Institute for Applied Mathematics
Winter term 2014/15
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"Markov Processes", Problem Sheet 5.

Hand in solutions before Friday 14.11, 2 pm
(post-box opposite to maths library)

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1. (Total variation distances)

a) Let $\nu = \bigotimes_{i=1}^{d} \nu_i$ and $\mu = \bigotimes_{i=1}^{d} \mu_i$ be two product probability measures on S^d. Show in at least two different ways that

$$\|\nu - \mu\|_{TV} \;\leq\; \sum_{i=1}^{d} \|\nu_i - \mu_i\|_{TV}.$$

b) Show that the total variation distance of the law of a Markov Chain to its stationary distribution is a non-increasing function of time.

c) Do similar statements as in a) and b) hold when the total variation distance is replaced by a general transportion metric $\mathcal{W}^1$?

2. (Hardcore model) Consider a finite graph (V, E) with n vertices of maximal degree Δ. The corresponding hardcore model with fugacity $\lambda > 0$ is the probability measure μ_λ on $\{0,1\}^V$ with mass function

$$\mu_\lambda(\eta) \;=\; \frac{1}{Z(\lambda)}\, \lambda^{\sum_{x\in V}\eta(x)} \quad \text{if } \eta(x)\cdot\eta(y) = 0 \text{ for any } (x,y)\in E, \quad \mu_\lambda(\eta) \;=\; 0 \quad \text{otherwise},$$

where $Z(\lambda)$ is a normalization constant.

a) Describe the transition rule for the Glauber dynamics with equilibrium μ_λ, and determine the transition probabilities.

b) Prove that for $\lambda < (\Delta - 1)^{-1}$ and $t \in \mathbb{N}$,

$$\mathcal{W}^1(\nu p^t, \mu) \;\leq\; \alpha(n,\Delta)^t\, \mathcal{W}^1(\nu,\mu) \;\leq\; \exp\left(-\frac{t}{n}\left(\frac{1-\lambda(\Delta-1)}{1+\lambda}\right)\right) \mathcal{W}^1(\nu,\mu),$$

where $\alpha(n,\Delta) = 1 - \frac{1}{n}\left(\frac{1-\lambda(\Delta-1)}{1+\lambda}\right)$, and $\mathcal{W}^1$ is the transportation metric based on the Hamming distance on $\{0,1\}^V$.

c) Show that the total variation mixing time is of order $O(n \log n)$.

3. (Stability of Markov chains on $\mathbb{R}^1$) Let (X_n, P_x) be a Markov chain on $\mathbb{R}^1$ with transition step $x \mapsto x + W_x$ where W_x is a random variable that depends on x in a measurable way. We assume that there exists a finite constant $r > 0$ such that $|W_x| \leq r$ for any $x \in \mathbb{R}$. Let

$$m(x) \;=\; E[W_x] \qquad \text{and} \qquad v(x) \;=\; E[W_x^2].$$

a) Prove that if $m(x) \leq \theta v(x)/(2x)$ for some $\theta < 1$ then the interval $(-a, a)$ is Harris recurrent for sufficiently large a.

   Hint: Consider the Lyapunov function $V(x) = \log(1+x)$, *and apply the Taylor expansion* $\log(1+x+w) = \log(1+x) + \frac{w}{1+x} - \frac{w^2}{2(1+x)^2} + O(|x|^{-3})$.

b) Prove that if $m(x) \geq \theta v(x)/(2x)$ for some $\theta > 1$ then the interval $(-a, a)$ is not recurrent for any $a > 0$.

   Hint: Consider $V(x) = 1 - (1+x)^{-\alpha}$.

c) Give a sufficient condition for geometric ergodicity of the Markov chain.