Institute for Applied Mathematics
Winter term 2014/15
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"Markov Processes", Problem Sheet 4.

Hand in solutions before Friday 7.11, 2 pm
(post-box opposite to maths library)

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1. (Couplings on $\mathbb{R}^d$) Let $W : \Omega \to \mathbb{R}^d$ be a random variable on $(\Omega, \mathcal{A}, P)$, and let μ_a denote the law of $a + W$.

a) (Synchronous coupling) Let $X = a + W$ and $Y = b + W$ for $a, b \in \mathbb{R}^d$. Show that

$$\mathcal{W}^2(\mu_a, \mu_b) \;=\; |a-b| \;=\; E(|X-Y|^2)^{1/2},$$

i.e., (X, Y) is an optimal coupling w.r.t. $\mathcal{W}^2$.

b) (Reflection coupling) Let $\widetilde{Y} = \widetilde{W} + b$ where $\widetilde{W} \equiv W - 2e \cdot W\, e$ with $e = \frac{a-b}{|a-b|}$. Prove that $(X, \tilde{Y})$ is also a coupling of μ_a and μ_b, and if $|W| \leq \frac{|a-b|}{2}$ a.s. then

$$E\left(f(|X-\widetilde{Y}|\right) \;\leq\; f(|a-b|) \;=\; E\left(f(|X-Y|)\right)$$

for any concave, increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$.

2. (Lazy random walk on the hypercube)
For $d \in \mathbb{N}$ consider the random walk on $\{0,1\}^d$ with initial distribution ν, where the transition rates p are constructed as follows: The random walk chooses uniformly at random one of its nearest neighbors and then jumps (given that choice) with probability $\frac{1}{2}$ to that new site and stays with probability $\frac{1}{2}$ at the current site.
Let μ denote the uniform distribution on the hypercube $\{0,1\}^d$. Prove that, for $n \geq 1$,

$$\mathcal{W}^1(\nu p^n, \mu) \leq \left(1 - \frac{1}{d}\right)^n \mathcal{W}^1(\nu, \mu).$$

3. (Tightness) Prove the following three statements.

a) A sequence of probability measures on the line is tight if and only if , for the corresponding distribution functions, we have $\lim_{x\to\infty} F_n(x) = 1$ and $\lim_{x\to-\infty} F_n(x) = 0$ uniformly in n.

b) A sequence of normal distributions on the line is tight if and only if the means and the variances are bounded (a normal distribution with variance 0 being a point mass).

c) A sequence of distributions of random variables X_n is tight if it is uniformly integrable.
Reminder: A sequence of random variables X_n is uniformly integrable if

$$\sup_{n\in\mathbb{N}} E[|X_n|; |X_n| \geq c] \to 0 \text{ as } c \to \infty.$$

4. (Contraction coefficients) Suppose that p is a Feller transition kernel on a complete separable metric space (S, d) satisfying the Lipschitz condition

$$\mathcal{W}^1(p(x,\cdot), p(y,\cdot)) \ \leq \ \alpha \cdot d(x,y) \qquad \text{for any } x, y \in S.$$

Prove that the bound

$$\mathcal{W}^1(\mu p, \nu p) \ \leq \ \alpha \cdot \mathcal{W}^1(\mu, \nu)$$

holds

a) for Dirac measures $\mu = \delta_x$, $\nu = \delta_y$,

b) for finite convex combinations of Dirac measures,

c) for arbitary probability measures μ, ν in $\mathcal{P}^1(S)$.

Hence conclude that the Wasserstein contraction coefficient $\alpha(p)$ coincides with the $\mathcal{W}^1$-Lipschitz norm of the transition kernel:

$$\alpha(p) \ = \ \sup_{x \neq y} \frac{\mathcal{W}^1(p(x,\cdot), p(y,\cdot))}{d(x,y)}.$$

Information from the Fachschaft
On 18th November, beginning at 18 o'clock the student council will host a plenary assembly for every math student. These Topics will be discussed: interim mensa, improvements of examination regulations and local numerus clausus. Further information on these topics are available at the showcase in the auxiliary building as well as on fsmath.uni-bonn.de . Attend numerously!