Institute for Applied Mathematics
Winter term 2014/15
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"Markov Processes", Problem Sheet 3.

Hand in solutions before Friday 31.10, 2 pm
(post-box opposite to maths library)

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1. (Strong Markov property and Harris recurrence) Let (X_n, P_x) be a time homogeneous Markov chain on the state space $(S, \mathcal{B})$ with transition kernel $p(x, dy)$.

a) Show that if T is a finite $(\mathcal{F}_n^X)$ stopping time, then conditionally given $\mathcal{F}_T^X$, the process $\hat{X}_n := X_{T+n}$ is a Markov chain with transition kernel p starting in X_T.

b) Conclude that a set $A \in \mathcal{B}$ is Harris recurrent if and only if

$$P_x(X_n \in A \text{ infinitely often}) = 1 \quad \text{for any } x \in A.$$

2. (Recurrence on discrete state spaces) Let (X_n, P_x) be an irreducible homogeneous Markov chain with countable state space S.

a) Prove that the following three conditions are equivalent:

   i) There exists a finite set $A \subset S$ such that A is recurrent.

   ii) $\{x\}$ is recurrent for any $x \in S$.

   iii) $P_x(X_n = y \text{ infinitely often}) = 1 \ \forall\, x, y \in S$.

b) Prove the equivalence of the following three conditions:

   i) There exists a finite set $A \subset S$ such that A is positive recurrent.

   ii) $\{x\}$ is positive recurrent for any $x \in S$.

   iii) $E_x(T_y) < \infty \ \forall\, x, y \in S$.

c) Show using Lyapunov functions that the simple random walk on $\mathbb{Z}^2$ is recurrent. (Hint: Consider for example the functions $v(x) = (\log(x))^\alpha$.)

3. (Lyapunov functions and stochastic stability)

a) Consider a perturbed random walk on $\mathbb{Z}^d$ with transition rates

$$p(x,y) = \begin{cases} \frac{1}{2d} + \delta(x,y) & \text{for } |x-y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find an estimate on the exit time from a ball of radius R. To this end consider for $\sigma > 0$ the function $F(x) = \exp\left(\sigma \sum_{i=1}^d |x_i|\right)$ on $\mathbb{Z}^d$ and show that

$$(pF)(x_1, \dots, x_d) \geq \theta F(x_1, \dots, x_d)$$

for some choices of $\sigma > 0$ and $\theta > 1$ that may depend on R.

b) Consider a state space model on $\mathbb{R}^d$ with one-step transition $x \to x + b(x) + \sigma(x)W$, where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable functions, and $W : \Omega \to \mathbb{R}^d$ is a random variable such that $E(W) = 0$ and $\mathrm{Cov}(W_i, W_j) = \delta_{ij}$. Show that sufficiently large balls are positive recurrent provided

$$\limsup_{|x|\to\infty} \left(2x \cdot b(x) + |b(x)|^2 + \mathrm{tr}(\sigma^T(x)\sigma(x))\right) \;<\; 0.$$

4. (Simulation of state space models) Write a routine (e.g. in Mathematica) for simulating and visualizing general state space models on $\mathbb{R}^2$. Experiment with different state space models and different parameters, e.g. you may compare the models in Exercise 3 b) in the regime where the condition for positive recurrence is satisfied and outside of that regime.