Institute for Applied Mathematics
Winter term 2014/15
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“Markov Processes”, Problem Sheet 2.

Hand in solutions before Monday 20.10., 2 pm (Exercises 1-3), resp. Friday 24.10, 2 pm (Ex. 4,5) (post-box opposite to maths library)

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1. (Conditional Expectations) Let X, Y, Z be random variables on $(\Omega, \mathcal{A}, P)$ such that X and Y are independent of Z. Prove that

$$E[X|\sigma(Y, Z)] \;=\; E[X|Y].$$

2. (Passage times of the simple random walk) Let $S_n = \sum_{i=1}^n Z_i$ where $(Z_n)_{n\geq 1}$ are independent random variables with $P(Z_n = 1) = P(Z_n = -1) = 1/2$. Let a be a strictly positive integer, and let $T_a = \inf\{n \geq 0 : S_n = a\}$ denote the first passage time of a.

a) Show that S_n and $S_n^2 - n$ are martingales. For $b < 0 < a$ compute $P[T_a < T_b]$ and $E[T_{\mathbb{Z}\setminus(a,b)}]$. Conclude that $E[T_a] = \infty$.

b) Show that for any $\theta \in \mathbb{R}$,
$$X_n^\theta \;=\; e^{\theta S_n}/(\cosh\theta)^n$$
is a martingale, and that for $\theta \geq 0$, $\left(X_{n\wedge T_a}^\theta\right)_{n\geq 0}$ is a bounded martingale that converges almost surely and in L^2 to the random variable
$$W^\theta \;=\; (\cosh\theta)^{-T_a} e^{\theta a}\, 1_{\{T_a<\infty\}}\,.$$
Conclude that $P(T_a < \infty) = 1$ and $E((\cosh\theta)^{-T_a}) = e^{-\theta a}$.

c) Explain how the results derived above can also be deduced from Corollary 1.7.

3. (Recurrence of Brownian motion) A continuous-time stochastic process $((B_t)_{t\in[0,\infty)}, P_x)$ taking values in $\mathbb{R}^d$ is called a *Brownian motion starting at x* if the sample paths $t \mapsto B_t(\omega)$ are continuous, $B_0 = x$ P_x-a.s., and for every $f \in C_b^2(\mathbb{R}^d)$, the process

$$M_t^{[f]} \;:=\; f(B_t) - \frac{1}{2}\int_0^t \Delta f(B_s)\, ds$$

is a martingale w.r.t. the filtration $\mathcal{F}_t^B = \sigma(B_s : s \in [0,t])$. Let $T_a = \inf\{t \geq 0 : |B_t| = a\}$.

a) Compute $P_x[T_a < T_b]$ for $a < |x| < b$.

b) Show that for $d \leq 2$, a Brownian motion is recurrent in the sense that $P_x[T_a < \infty] = 1$ for any $a < |x|$.

c) Show that for $d \geq 3$, a Brownian motion is transient in the sense that $P_x[T_a < \infty] \to 0$ as $|x| \to \infty$.

You may assume the optional stopping theorem and the martingale convergence theorem in continuous time without proof. You may also assume that the Laplacian applied to a rotationally symmetric function $g(x) = \gamma(|x|)$ is given by

$$\Delta g(x) \;=\; r^{1-d}\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}\gamma\right)(r) \;=\; \frac{d^2}{dr^2}\gamma(r) + \frac{d-1}{r}\frac{d}{dr}\gamma(r) \qquad \textit{where } r = |x|.$$

(How can you derive this expression rapidly if you do not remember it ?)

4. (Random walks on $\mathbb{Z}$) Let $((X_n)_{n\geq 0}, (P_x)_{x\in\mathbb{Z}})$ be the canonical Markov chain on $\mathbb{Z}$ with transition matrix Q given by

$$Q(x, x+1) = p,\ Q(x,x) = r,\ Q(x, x-1) = q$$

where $p + q + r = 1, p > 0, q > 0, r \geq 0$. Fix $a, b \in \mathbb{Z}$ with $a < b - 1$ and let $T = \inf\{n \geq 0 : X_n \notin (a,b)\}$.

a) Prove that for any function $g : \{a+1, a+2, \ldots, b-1\} \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, the system

$$\begin{aligned} &(Q - I)u(x) = -g(x), \quad a < x < b, \\ &u(a) = \alpha,\ u(b) = \beta, \end{aligned} \tag{1}$$

has a unique solution.

b) Conclude that $E_x(T) < \infty$ for any x. How can the mean exit time be computed explicitly ?

c) Assume for the moment that for every $s > 0$ and $x \in$, $u_s(x) := E_x(T^s) < \infty$. Prove that u_2 is a solution of (1) for some α, β, g to be determined as functions of u_1.

d) Prove that there exists $\epsilon > 0$ such that $E_x[\exp(\lambda T)] < \infty$ for any $\lambda < \epsilon$. Hence conclude that $E_x(T^s) < \infty$ for every $s > 0$.

5. (Feynman-Kac formula) This exercise gives a direct proof of the uniqueness part in the Feynman-Kac formula. Let $((X_n)_{n\geq 0}, P_x)$ be a canonical time-homogeneous Markov chain with generator $\mathcal{L}$ on the state space S. Let $w : S \to \mathbb{R}_+$ be a nonnegative function.

a) For which functions v is

$$M_n = e^{-\sum_{k=0}^{n-1} w(X_i)} v(X_n)$$

a martingale?

b) Let $D \subset S$ be a measurable subset such that $T = \inf\{n > 0 : X_n \in D^c\} < \infty$ P_x-a.s. for any x, and let v be a bounded solution to the boundary value problem

$$\begin{aligned} &(\mathcal{L}v)(x) = (e^{w(x)} - 1)u(v) \quad \forall x \in D, \\ &v(x) = f(x) \quad \forall x \in D^c. \end{aligned} \tag{2}$$

Show by using a) that

$$v(x) \;=\; E_x\left(e^{-\sum_{k=0}^{T-1} w(X_k)} f(X_T)\right).$$