Markov Processes SS 2008 Series 6

1. (Total variation distance). Let P, Q be two probability measures on a measurable space S, and let d_{TV} denote the total variation distance on the set of probability measures on S. Prove that

$$d_{TV}(P,Q) = \frac{1}{2} \cdot \sup_{f\colon \|f\|_{\sup} \le 1} \Big| \int_S f\,dP - \int_S f\,dQ \Big| \,,$$

where the supremum is over all measurable functions $f : S \to \mathbb{R}$ such that $\|f\|_{\sup} := \sup_x |f(x)| \le 1$.

2. (Hypercontractivity implies log Sobolev). State the equivalence between logarithmic Sobolev inequalities and hypercontractivity. Prove that hypercontractivity implies the LSI.

3. (Stationary distributions). Let $\{X_t,\, t \ge 0\}$ be a real-valued regular time-homogeneous diffusion process with the drift parameter $\mu(x)$ and the diffusion parameter $\sigma^2(x)$. Suppose that the process X has at least one stationary density ψ.

a) Give an explicit expression for ψ in terms of μ and σ^2.

b) Using this expression, formulate a necessary condition for existence of a stationary distribution of X.

c) In the Wright-Fischer model with mutation one has $\sigma^2(x) = x(1-x)$ and $\mu(x) = -\alpha_1 x + \alpha_2(1-x)$, where α_1, α_2 are some real parameters. For what values of α_1 and α_2 this process has no stationary densities?

4. (Logarithmic Sobolev inequality on two points). Consider the two-point space $S = \{-1, 1\}$ with the Bernoulli measure μ which assigns weight $1/2$ to each point, and the transition probability function

$$p_t(x,y) = \begin{cases} \frac{1+e^{-t}}{2}, & \text{if x = y;} \\ \frac{1-e^{-t}}{2}, & \text{if x = -y.} \end{cases}$$

Let $\mathcal{E}$ be the Dirichlet form associated with $p_t(x,\cdot)$ and μ.

a) Prove that for any bounded measurable $f : S \to \mathbb{R}$

$$(1) \qquad \int_S f^2 \log \frac{|f|^2}{\|f\|^2_{L^2(\mu)}} \, d\mu \le 2\,\mathcal{E}(f, f)$$

b) Conclude from this that the associated semigroup $\{P_t,\, t > 0\}$ has the property that $\|P_t\|_{L^p(\mu)\to L^q(\mu)} = 1$, as long as $1 < p < q < \infty$ and $e^{2t} \ge (q-1)/(p-1)$.

c) Show that the constant in (1) is optimal.

Hint: First observe that it suffices to prove (1) for f of the form $f_b(x) = 1 + bx$ where $b \in [0,1]$. Then show that (1) for f_b is equivalent to

$$h(b) \equiv (1+b)^2 \log(1+b) + (1-b)^2 \log(1-b) - (1+b^2)\log(1+b^2) \le 2b^2$$

for $b \in [0,1]$. Finally, prove the preceding by checking that $h(0) = h'(0) = 0$ and that $h''(b) \le 4$.