Markov Processes SS 2008 Series 5

1. (Martingales of compound Poisson processes). Let $Z_1, Z_2, \ldots$ be independent identically distributed $\mathbb{R}^d$-valued random variables. Let K_t be a Poisson process of intensity λ that is independent of the Z_i. Consider the compound Poisson process $(X_t)_{t \geq 0}$ defined by

$$X_t = \sum_{i=1}^{K_t} Z_i \,.$$

Prove that the following processes are martingales:

a) $M_t = X_t - b \cdot t$, where $b := \lambda \cdot E\left[Z_1\right]$, provided $E\left[|Z_1|\right] < \infty$.

b) $|M_t|^2 - a \cdot t$, where $a := \lambda \cdot E\left[|Z_1|^2\right]$, provided $E\left[|Z_1|^2\right] < \infty$.

c) $\exp\Big(ip{\cdot}X_t{+}\psi(p)\, t\Big)$, where $p \in \mathbb{R}^d$ and ψ is the characteristic exponent.

2. (Poisson point processes). Give an explicit construction (with complete proofs) of a Poisson point process with finite intensity ν.

3. (Transformations of Lévy processes).

a) State without proof the Lévy-Khinchin representation for Lévy processes.

b) Show that if X_t is a Lévy process then the processes $-X_t$ and $X_t + ct$, $c \in \mathbb{R}$, are again Lévy processes.

c) How are the characteristics in the Lévy-Khinchin representation of these processes related to those for X_t ?

4. (Stable processes). State the Lévy-Itô representation for α-stable processes, $\alpha \in (0, 2]$. Show that an α-stable process has finite mean if and only if $\alpha > 1$.

5. (Olber's paradox). Suppose that stars occur in $\mathbb{R}^3$ at the points R_i, $i \in \mathbb{N}$, of a spatial Poisson process with intensity λ. The star at R_i has brightness B_i, where the B_i are i.i.d. with mean β. The total illumination at the origin from stars within a large ball with radius a is

$$I_a \;=\; \sum_{i:|R_i|\le a} \frac{cB_i}{|R_i|^2}$$

for some absolute constant c. Show that

$$E[I_a] \;=\; 4\pi\lambda c\beta a\,.$$

The fact that this is unbounded as $a \to \infty$ is called *Olber's paradox*, and suggests that the celestial sphere should be uniformly bright at night. The fact that it is not is a problem whose resolution is still a matter for debate. One plausible explanation relies on a sufficiently fast rate of expansion of the Universe.

6. (Semigroups, generators and resolvents). Consider a strongly continuous contraction semigroup $(P_t)_{t\ge 0}$ on a Banach space B. For $\alpha > 0$, the corresponding *α-resolvent* is the linear operator $G_\alpha : B \to B$ defined by

$$G_\alpha f \;=\; \int_0^\infty e^{-\alpha t}\, P_t f\, dt.$$

a) Prove that each G_α is a contraction.

b) Show that for all $f \in B$, $G_\alpha f$ is contained in the domain of the generator L of P_t, and
$$LG_\alpha f \;=\; f - \alpha G_\alpha f.$$
Conclude that
$$G_\alpha \;=\; (\alpha - L)^{-1}.$$

c) Show that $\alpha G_\alpha f$ converges to f as $\alpha \to \infty$. Conclude that the generator of a C_0 contraction semigroup is densely defined.