Markov Processes SS 2008 Series 4

1. (Infinitesimal characterization of stationary distributions). Let p_t, $t \geq 0$, be the transition semigroup of a continuous time-homogeneous Markov chain on a finite state space S.

a) Show that μ is a stationary distribution if and only if

$$\sum_{x \in S} \mu(x)\, \mathcal{L}(x, y) \;=\; 0 \qquad \text{for all } y \in S,$$

where $\mathcal{L}$ is the generator of the process.

b) Show that the transition matrices are self-adjoint in $L^2(\mu)$, i.e.,

$$\sum_{x \in S} f(x)\, (p_t g)(x)\; \mu(x) \;=\; \sum_{x \in S} (p_t f)(x)\, g(x)\; \mu(x) \qquad \forall\, t \geq 0,\; f, g : S \to \mathbb{R},$$

if and only if the generator satisfies the detailed balance condition w.r.t. μ. What does this mean for the process ?

2. (Stationary distributions of Feller semigroups). Let p_t, $t \geq 0$, be the transition semigroup of a time-homogeneous Markov process on a compact state space S. Suppose that p_t has the Feller property, and let $\mathcal{J}$ denote the set of all invariant probability measures. Prove:

a) $\mu \in \mathcal{J}$ if and only if

$$\int p_t f \, d\mu \;=\; \int f \, d\mu \qquad \text{for all } f \in C(S) \text{ and } t \geq 0.$$

b) $\mathcal{J}$ is as a compact subset of $\mathcal{M}_1(S)$.

c) If $\mu = \lim_{t \to \infty} \nu p_t$ exists for some $\nu \in \mathcal{M}_1(S)$, then $\mu \in \mathcal{J}$.

d) If $\mu = \lim_{n \to \infty} t_n^{-1} \int_0^{t_n} \nu p_t \, dt$ exists for some $\nu \in \mathcal{M}_1(S)$ and some $t_n \uparrow \infty$, then $\mu \in \mathcal{J}$.

e) $\mathcal{J}$ is not empty.

3. (Simple exclusion process). Let $\mathbb{Z}_n^d = \mathbb{Z}^d/(n\mathbb{Z})^d$ denote a discrete d-dimensional torus. The simple exclusion process on $S = \{0,1\}^{\mathbb{Z}_n^d}$ is the Markov process with generator

$$(\mathcal{L}f)(\eta) \;=\; \frac{1}{2d}\sum_{x\in\mathbb{Z}_n^d}\sum_{y:|y-x|=1} I_{\{\eta(x)=1,\eta(y)=0\}}\cdot\left(f(\eta^{x,y})-f(\eta)\right),$$

where $\eta^{x,y}$ is the configuration obtained from η by exchanging the values at x and y. Show that any Bernoulli measure of type

$$\mu_p \;=\; \bigotimes_{x\in\mathbb{Z}_n^d}\nu_p\,, \qquad \nu_p(1)=p,\;\nu_p(0)=1-p,$$

$p\in[0,1]$, is a stationary distribution. Why does this not contradict the fact that any irreducible Markov process on a finite state space has a unique stationary distribution ?

4. (Adjoint processes). Let p_t, $t\geq 0$, be the transition semigroup of a continuous time-homogeneous Markov chain on a finite state space S with generator $\mathcal{L}$. Let μ be a probability measure with full support on S.

a) Write down explicitly the adjoint $\mathcal{L}^*$ of $\mathcal{L}$ as an operator in $L^2(\mu)$. Prove that $\mathcal{L}^*$ is the generator of a Markov process if and only if μ is stationary w.r.t. $(p_t)_{t\geq 0}$.

b) Show that in this case, the Markov process generated by $\mathcal{L}^*$ has the transition semigroup p_t^*.

c) Give a probabilistic interpretation of this process when μ is the initial distribution.