Markov Processes SS 2008 Series 3

1. (A queueing model). Jobs arrive in a computer queue in the manner of a Poisson process with intensity λ. The central processor handles them one by one in the order of their arrival, and each has an exponentially distributed runtime with parameter μ, the runtimes of different jobs being independent of each other and of the arrival process. Let $X(t)$ be the number of jobs in the system (either waiting or running) at time t, where $X(0) = 0$.

a) Explain why X is a Markov chain, and write down its generator.

b) Show that a stationary distribution exists if and only if $\lambda < \mu$, and find it in this case.

2. (Stationarity versus reversibility).

a) Show that any stationary distribution of a birth-and-death chain satisfies the detailed balance condition.

b) Give examples of Markov processes whose stationary distributions do not satisfy the detailed balance condition.

3. (Multinomial resampling). Each of n particles in a population changes its type with rate 1 to a type that is randomly selected from the types of all n particles.

a) Write down the generator of the corresponding Markov process.

b) Show that the number of particles of a given type is a birth and death process, and identify the birth and death rates.

c) Find the stationary distribution of the birth and death process.

d) Determine a stationary distribution of the original Markov process when the type space is finite (or, if you prefer, when the type space consists of two elements).

4. (Decay of relative entropy). Consider a continuous time irreducible Markov chain on a finite state space S. Suppose there exists a probability distribution μ on S satisfying the detailed balance condition such that $\mu(x) > 0$ for all $x \in S$. Show that if ν_t denotes the distribution at time t of the process started with an arbitrary initial distribution ν_0, then the L^2 *distance*

$$\chi^2(\nu_t|\mu) \ := \ \sum_{x \in S} \left(\frac{\nu_t(x)}{\mu(x)} - 1 \right)^2 \mu(x)$$

and the *relative entropy*

$$H(\nu_t|\mu) \ := \ \sum_{x \in S} \frac{\nu_t(x)}{\mu(x)} \log \left(\frac{\nu_t(x)}{\mu(x)} \right) \mu(x)$$

are non-increasing functions of t.

5. (Pasta property). Let $X = \{X(t) : t \geq 0\}$ be a Markov chain having stationary distribution π. We may sample X at the times of a Poisson process: let N be a Poisson process with intensity λ, independent of X, and define $Y_n = X(T_n+)$, the value taken by X immediately after the epoch T_n of the nth arrival of N. Show that $Y = \{Y_n : n \geq 0\}$ is a discrete-time Markov chain with the same stationary distribution as X.