Markov Processes SS 2008 Series 2

1. (Two-state chain). Consider the Markov jump process with state space $E = \{0, 1\}$; transition matrix $\pi = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$, where $\alpha, \beta \in (0, 1)$; and constant intensity $\lambda > 0$ for the underlying Poisson process. Suppose that $X(0) = 0$.

a) Find the transition semigroup $\{\mathbf{P}(t)\}_{t \geq 0}$.

b) Determine the limiting distribution μ of the process as $t \to \infty$. Show that it solves $\mu\mathcal{L} = 0$ where $\mathcal{L}$ is the generator.

2. (Immigration-death process). Particles in a population die independently with rate $\mu > 0$. In addition, immigrants arrive with rate $\lambda > 0$. Assume that the population consists initially of one particle.

a) Explain why the population size X_t can be modeled by a birth-death process with rates $b(n) = \lambda$ and $d(n) = n\mu$.

b) Show that the generating function $G(s, t) = \mathbb{E}(s^{X(t)})$ is given by

$$G(s,t) = \{1 + (s-1)e^{-\mu t}\} \exp\Big\{\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})\Big\}$$

c) Deduce the limiting distribution of $X(t)$ as $t \to \infty$.

3. (Explosion). Let $(X_t)_{t \geq 0}$ be a Markov chain on the integers with transition rates

$$q_{i,i+1} = \lambda q_i, \quad q_{i,i-1} = \mu q_i$$

and $q_{i,j} = 0$ if $|j - i| \geq 2$, where $\lambda + \mu = 1$ and $q_i > 0$ for all i.

a) In the case where $\mu = 0$, write down a necessary and sufficient condition for $(X_t)_{t \geq 0}$ to be explosive.

b) Why is this condition necessary for $(X_t)_{t \geq 0}$ to be explosive for all $\mu \in [0, 1/2)$?

4. (Compound Poisson process). Let N be a Poisson process with constant intensity λ, and let $Y_1, Y_2, \ldots$ be independent random variables with common characteristic function ϕ and density function f. The process $N^*(t) = Y_1 + Y_2 + \ldots + Y_{N(t)}$ is called a *compound* Poisson process.

a) Write down a forward equation for N^*.

b) Find the characteristic function of $N^*(t)$.

5. (Time-dependent birth process). The Markov chain $\{X_t : t \geq 0\}$ is a birth process whose intensities $\lambda_k(t)$ depend also on the time t and are given by

$$\mathbb{P}\Big(X(t+h) = k+1 \,|\, X(t) = k\Big) = \frac{1+\mu k}{1+\mu t}\, h + o(h)$$

as $h \downarrow 0$. Suppose that $X(0) = 1$.

a) Show that the probability generating function $G(s,t) = \mathbb{E}(s^{X(t)})$ satisfies

$$\frac{\partial G}{\partial t} = \frac{s-1}{1+\mu t}\left\{G + \mu s\,\frac{\partial G}{\partial s}\right\}, \qquad 0 < s < 1.$$

b) Find the mean and variance of $X(t)$.