Markov Processes SS 2008 Series 1

1. Arrivals of the Number 1 bus form a Poisson process of rate one bus per hour, and arrivals of the Number 7 bus form an independent Poisson process of rate seven buses per hour.

a) What is the probability that exactly three buses pass by in one hour?

b) What is the probability that exactly three Number 7 buses pass by while I am waiting for a Number 1?

c) When the maintenance depot goes on strike half the buses break down before they reach my stop. What, then, is the probability that I wait for 30 minutes without seeing a single bus?

2. (Strong Markov property). Let $\{X_n\}$ be a discrete-time nonhomogeneous Markov chain, and let T be a stopping time that is finite with probability 1. Prove the strong Markov property for $\{X_n\}$.

3. (A branching process in continuous time). Each bacterium in a colony splits into two identical bacteria after an exponential time of parameter λ, which then split in the same way but independently. Let X_t denote the size of the colony at time t, and suppose $X_0 = 1$.

a) Show that the probability generating function $\phi(t) = \mathbb{E}\left(z^{X_t}\right)$ satisfies

$$\phi(t) = z\, e^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \phi(t-s)^2 \, ds \,.$$

b) Make a change of variables $u = t - s$ in the integral and deduce that $d\phi/dt = \lambda\phi(\phi - 1)$.

c) Deduce that, for $q = 1 - e^{-\lambda t}$ and $n = 1, 2, \ldots$

$$\mathbb{P}(X_t = n) = q^{n-1}(1-q).$$

4. (A jump process with explosion). Let $\{X_t \,|\, t \geq 0\}$ be a Markov jump process with state space $E = \mathbb{Z} \cup \{\Delta\}$, where $\Delta \notin \mathbb{Z}$ is some extra state (needed to describe the process after possible explosion). Let the intensities of the process be $\lambda(k) = (k+1)^2$ and the jump probabilities be given by $p_{k,k+1} = 1$ for all $k \in \mathbb{Z}$.

a) Show that the process is explosive.

b) The state s is called *instantaneous* if the process jumps out of s immediately after s has been entered. Show that there exists a version of $\{X_t\}_{t\geq 0}$ with Δ as instantaneous state and $X_t \to -\infty$ as $t \downarrow \zeta$.

5. A pedestrian wishes to cross a single lane of fast-moving traffic. Suppose the number of vehicles that have passed by time t is a Poisson process of rate λ, and suppose it takes time a to walk across the lane. Assume that the pedestrian can foresee correctly the times at which vehicles will pass by and that the pedestrian will not cross the lane if, at any time whilst crossing, a car would pass in either direction.

a) How long on average does it take to cross over safely?

How long on average does it take to cross two similar lanes

b) when one must walk straight across?

c) when an island in the middle of the road makes it safe to stop halfway?

6. (Doubly stochastic Poisson process). If the intensity function $\lambda(t)$ of a non-homogeneous Poisson process N is itself a random process, then N is called a *doubly stochastic* Poisson process. Consider the case when $\lambda(t) = \Lambda$ for all t, and Λ is a random variable taking either of two values λ_1 or λ_2, each being picked with equal probability 1/2. Find $\mathbb{E}[N(t)]$ and $Var(N(t))$.