

„Introduction to Stochastic Analysis” Problem Sheet 9

Please hand in your solutions before 12 noon on Tuesday, December 11.

1. (Solutions of SDE).

Let (B_t) be a one-dimensional Brownian motion with $B_0 = 0$. Show that the following processes solve the corresponding stochastic differential equations:

a) $X_t = B_t/(1+t)$ solves

$$dX_t = -(1+t)^{-1}X_t dt + (1+t)^{-1} dB_t, \quad X_0 = 0.$$

b) $X_t = \sin(B_t)$ solves

$$dX_t = -\frac{1}{2}X_t dt + \sqrt{1-X_t^2} dB_t, \quad X_0 = 0,$$

for $t < \inf \{s > 0 : B_s \notin [-\pi/2, \pi/2]\}$.

c) $(X_t, Y_t) = (t, e^t B_t)$ solves

$$\begin{bmatrix} dX_t \\ dY_t \end{bmatrix} = \begin{bmatrix} 1 \\ Y_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_t} \end{bmatrix} dB_t, \quad \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

d) $(X_t, Y_t) = (\cosh(B_t), \sinh(B_t))$ solves

$$\begin{bmatrix} dX_t \\ dY_t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X_t \\ Y_t \end{bmatrix} dt + \begin{bmatrix} Y_t \\ X_t \end{bmatrix} dB_t, \quad \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

e) For $x \geq 0$, $X_t = (x^{1/3} + \frac{1}{3}B_t)^3$ solves

$$dX_t = \frac{1}{3}X_t^{1/3} dt + X_t^{2/3} dB_t, \quad X_0 = x.$$

Is the solution of this SDE unique?

2. (Martingales of Brownian motion).

Let (B_t) be a one-dimensional Brownian motion with $B_0 = 0$. Show that the following processes are martingales:

a) $X_t = e^{\frac{1}{2}t} \cos(B_t)$, b) $X_t = e^{\frac{1}{2}t} \sin(B_t)$, c) $X_t = (B_t + t) \exp(-B_t - \frac{1}{2}t)$.

3. (Random rotations: Itô vs. Stratonovich).

We consider stochastic differential equations of the form

$$dZ_t = A Z_t dB_t, \quad Z_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (1)$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the antisymmetric matrix generating the unit rotation in \mathbb{R}^2 , (B_t) is a *one dimensional* Brownian motion, and the solution (Z_t) is a stochastic process taking values in \mathbb{R}^2 .

- Write down a time-discretization of the Itô equation (1), and simulate sample paths of the solution.
- What do you observe? Can you explain your observations?
- Now consider the Stratonovich equation

$$\circ dZ_t = A Z_t \circ dB_t, \quad Z_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (2)$$

Find a numerical discretization for the SDE and simulate approximate solutions. What do you observe now?

Hint: Make sure that before starting the implementation, you have transformed the discretization into an accessible form.

Matrix inversion in Python: `from scipy import linalg; inversematrix=linalg.inv(matrix)`

4. (*optional: Quadratic variation of Brownian motion revisited).

Let $(B_t)_{t \geq 0}$ be a Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The goal of this exercise is to show that for an *arbitrary* sequence of partitions such that $\pi_n \subset \pi_{n+1}$ and $\text{mesh}(\pi_n) \rightarrow 0$, the quadratic variation $[B]_t$ exists almost surely. W.l.o.g., we assume $B_0 = 0$.

- Show that if $h : [0, \infty) \rightarrow \{-1, 1\}$ is a measurable deterministic function, then the process $I_t^h := \int_0^t h(s) dB_s$ is again a Brownian motion. Conclude that for any $0 \leq s \leq s'$, the process (\tilde{B}_t) defined by

$$\tilde{B}_t := \begin{cases} B_t & \text{for } t \in [0, s], \\ B_s - (B_t - B_s) & \text{for } t \in [s, s'], \\ B_s - (B_{s'} - B_s) + B_t - B_{s'} & \text{for } t \in [s', \infty), \end{cases}$$

is a Brownian motion, i.e., $(\tilde{B}_t) \sim (B_t)$.

- Now fix $t \geq 0$, and let \mathcal{F}_n denote the σ -algebra generated by the random variables $(B_{s' \wedge t} - B_{s \wedge t})^2$, where s and s' are successive partition points in π_m for some $m \geq n$. Show that

$$\sum (B_{s' \wedge t} - B_{s \wedge t})^2 = \mathbb{E} \left[\sum (B_{s' \wedge t} - B_{s \wedge t})^2 \middle| \mathcal{F}_n \right] = E [B_t^2 | \mathcal{F}_n],$$

where the sum is over all partition points $s \in \pi_n$.

- Conclude that $\sum (B_{s' \wedge t} - B_{s \wedge t})^2$ converges almost surely as $n \rightarrow \infty$, and identify the limit.