Institut für angewandte Mathematik
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UNIVERSITÄT BONN

„Introduction to Stochastic Analysis"
Problem Sheet 8

Please hand in your solutions before 12 noon on Tuesday, December 4.

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1. (Time-dependent Itô formula). Suppose that $X : [0, \infty) \to \mathbb{R}$ is a continuous function with continuous quadratic variation $[X]$ w.r.t. a sequence $(\pi_n)_{n\in\mathbb{N}}$ of partitions s.t. $\mathrm{mesh}(\pi_n) \to 0$. Show that for every function $F \in C^2(\mathbb{R}^2)$ and for every $t \in \mathbb{R}_+$, the Itô integral

$$\int_0^t \frac{\partial F}{\partial x}(s, X_s)\, dX_s \;=\; \lim_{n\to\infty} \sum_{s\in\pi_n} \frac{\partial F}{\partial x}(s, X_s)\,(X_{s'\wedge t} - X_{s\wedge t})$$

exists, and the time-dependent Itô formula

$$F(t, X_t) - F(0, X_0) \;=\; \int_0^t \frac{\partial F}{\partial s}(s, X_s)\, ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s)\, dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(X_s)\, d[X]_s \quad (1)$$

holds.

Hint: You may assume without proof that by Taylor's formula, there exists a function $o : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* $o(r)/r \to 0$ *as* $r \to 0$*, such that for any* $s, s' \in [0, t]$,

$$\begin{aligned} F(s', X_{s'}) - F(s, X_s) \;=\;& \frac{\partial F}{\partial s}(s, X_s)\,\delta s + \frac{\partial F}{\partial x}(s, X_s)\,\delta X_s + \frac{1}{2}\frac{\partial^2 F}{\partial s^2}(s, X_s)\,(\delta s)^2 \\ &+ \frac{\partial^2 F}{\partial s \partial x}(s, X_s)\,\delta s\,\delta X_s + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(s, X_s)\,(\delta X_s)^2 + o\left((\delta s)^2 + (\delta X_s)^2\right). \end{aligned}$$

2. (Geometric Brownian motion). Let $(B_s)_{s\ge 0}$ be an $(\mathcal{F}_s)$–Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$ s.t. $B_0 = 0$. A geometric Brownian motion $(X_s)_{s\ge 0}$ with parameters $\mu, \alpha \in \mathbb{R}$ is a solution of the stochastic differential equation (SDE)

$$dX_t \;=\; \mu\, X_t\, dt + \alpha\, X_t\, dB_t\,,$$

i.e., $(X_s)_{s\ge 0}$ is an almost surely continuous and $(\mathcal{F}_s)$ adapted process such that $\mathbb{P}$-almost surely,

$$X_t - X_0 \;=\; \mu \int_0^t X_s\, ds + \alpha \int_0^t X_s\, dB_s \qquad \text{for any } t \ge 0.$$

a) Find a solution of the SDE with initial value $X_0 = x_0$ using the ansatz

$$X_t \;=\; x_0 \cdot \exp(aB_t + bt)\,.$$

Here you may assume the time-dependent Itô fomula (1).

b) What can you say about the asymptotic behavior of the process as $t \to \infty$?

c) Compute $E[X_t]$ and $\operatorname{Cov}[X_s, X_t]$ for $s, t \geq 0$.

3. (Stochastic integrals w.r.t. Itô processes). Let

$$I_s \ := \ \int_0^s H_r \, dB_r, \qquad 0 \leq s \leq t,$$

with an $(\mathcal{F}_s)$–Brownian motion B on $(\Omega, \mathcal{A}, \mathbb{P})$, and a process $H \in \mathcal{L}_a^2(0, t; B)$. Suppose that $(\pi_n)_{n \in \mathbb{N}}$ is a sequence of partitions of $[0, t]$ such that $\operatorname{mesh}(\pi_n) \to 0$.
Prove that if G is an $(\mathcal{F}_s)$–adapted bounded continuous process, then the Riemann sums $\sum_{s \in \pi_n} G_s \cdot (I_{s'} - I_s)$ converge in $L^2(\mathbb{P})$, and

$$\int_0^t G_s \, dI_s \ = \ \lim_{n \to \infty} \sum_{s \in \pi_n} G_s \cdot (I_{s'} - I_s) \ = \ \int_0^t G_s \, H_s \, dB_s \ .$$

Hint: Express the Riemann sums as a stochastic integral $\int_0^t \ldots \, dB_s$ w.r.t. Brownian motion.

4. (A local martingale that is not a martingale). Let $(B_t)_{t \geq 0}$ be a Brownian motion in $\mathbb{R}^3$ with initial value $B_0 = x$, $x \neq 0$. Show that:

a) $X_t = 1/\|B_t\|$ is a local martingale up to $T = \inf\{t \geq 0 : B_t = 0\}$.

b) $T = \infty$ almost surely.

c) $\{X_s : 0 \leq s \leq t\}$ is uniformly integrable for all $t \geq 0$.

d) X_t is *not* a martingale.

Hint: You may assume without proof the multi-dimensional Itô formula for Brownian motion: If U is an open subset of $\mathbb{R}^d$, then for $F \in C^2(U)$,

$$F(B_t) - F(B_0) \ = \ \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x^i}(B_s) \, dB_s^i + \frac{1}{2} \int_0^t \Delta F(B_s) \, ds \qquad \forall \ t < T_{U^C},$$

where $T_{U^C} = \inf\{t \geq 0 : B_t \notin U\}$ is the first exit time from U.