Institut für angewandte Mathematik
Wintersemester 2018/19
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UNIVERSITÄT BONN

„Introduction to Stochastic Analysis"
Problem Sheet 7

Please hand in your solutions before 12 noon on Tuesday, November 27.

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1. (Local martingales). Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Show that if $(M_t)_{t\in[0,\infty)}$ is an $(\mathcal{F}_t)$ martingale on $(\Omega, \mathcal{A}, \mathbb{P})$ and $T : \Omega \to [0, \infty]$ is predictable, then $(M_t)_{t\in[0,T)}$ is a local martingale up to T.

b) Conversely, suppose that (M_t) is a local martingale up to $T = \infty$. Show that (M_t) is a martingale if and only if there exists a localizing sequence $(T_k)_{k\in\mathbb{N}}$ such that for every $t \in [0, \infty)$, the family of random variables $\{M_{t\wedge T_k} : k \in \mathbb{N}\}$ is uniformly integrable.

2. (Uniqueness of the angle bracket process). Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Suppose that (M_t) is a square integrable continuous $(\mathcal{F}_t)$ martingale with paths of almost surely bounded variation, i.e., for every $t \in \mathbb{R}_+$, the first variation

$$V_t^{(1)}(M) \;=\; \sup_{\pi} \sum_{s\in\pi} |M_{s'\wedge t} - M_{s\wedge t}|$$

is an almost surely bounded random variable. Show that $t \mapsto M_t$ is almost surely constant.
Hint: $\mathbb{E}[(M_t - M_0)^2] = \sum_{s\in\pi} \mathbb{E}[(M_{s'\wedge t} - M_{s\wedge t})^2]$.

b) More generally, prove that a continuous local martingale M with almost surely finite variation paths is almost surely constant.

c) Conclude that the angle bracket process $\langle M\rangle$ of a continuous local martingale is uniquely determined up to modification on a measure zero set.

3. (Progressive measurability). Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration.

a) Prove that an $(\mathcal{F}_t)$ adapted left-continuous stochastic process $(X_t)_{t\in[0,\infty)}$ is $(\mathcal{F}_t)$ progressively measurable.

b) Show that if $(X_t)_{t\in[0,\infty)}$ is a progressively measurable process and $T : \Omega \to [0, \infty]$ is an $(\mathcal{F}_t)$ stopping time then the random variable $X_T \cdot 1_{T<\infty}$ is measurable w.r.t. $\mathcal{F}_T$.

On December 3rd 7 pm (st) in the Lipschitz hall the Fachschaft (student council) organizes a Ladies Night for all female students enrolled in the Bachelor's program for mathematics (5th semester or higher) or in the Master's program. In case of questions, you can contact us via gleichstellung@fsmath.uni-bonn.de.