Institut für angewandte Mathematik
Wintersemester 2018/19
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„Introduction to Stochastic Analysis"
Problem Sheet 6

Please hand in your solutions before 12 noon on Tuesday, November 20.

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1. (Itô and Stratonovich integrals).
Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion starting at 0 on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let $\pi_n = \{k2^{-n} : k \in \mathbb{Z}_+\}$ be the n-th dyadic partition of $\mathbb{R}_+$.

a) Show that for any $t \geq 0$,

$$\lim_{n\to\infty} \sum_{s\in\pi_n} (B_{s'\wedge t} - B_{s\wedge t})^2 \;=\; \lim_{n\to\infty} \sum_{s\in\pi_n} (s' \wedge t - s \wedge t) \;=\; t \quad \text{in } L^2(\mathbb{P}),$$

and for $p > 2$,

$$\lim_{n\to\infty} \sum_{s\in\pi_n} |B_{s'\wedge t} - B_{s\wedge t}|^p \;=\; 0 \quad \text{in } L^2(\mathbb{P}).$$

b) The Stratonovich integral of a process (H_s) w.r.t. (B_s) over $[0,t]$ is defined by

$$\int_0^t H_s \circ dB_s \;:=\; \lim_{n\to\infty} \sum_{s\in\pi_n} \frac{1}{2}(H_{s'\wedge t} + H_{s\wedge t}) \cdot (B_{s'\wedge t} - B_{s\wedge t})$$

if the limit exists in $L^2(\mathbb{P})$. Show that

$$\int_0^t B_s \circ dB_s \;=\; \frac{1}{2}B_t^2 \qquad \text{and} \qquad \int_0^t B_s \, dB_s \;=\; \frac{1}{2}B_t^2 - \frac{1}{2}t\,.$$

c) More generally, prove that for every $m \in \mathbb{N} \setminus \{1\}$,

$$B_t^m \;=\; m\int_0^t B_s^{m-1} \circ dB_s \;=\; m\int_0^t B_s^{m-1}\, dB_s \;+\; \frac{1}{2}m(m-1)\int_0^t B_s^{m-2}\, ds.$$

Hint: You can use the identities

$$\begin{aligned}(x+h)^m - x^m \;&=\; mx^{m-1}h + \binom{m}{2}x^{m-2}h^2 + O(|h|^3),\\ (x+h/2)^m - (x-h/2)^m \;&=\; mx^{m-1}h + O(|h|^3).\end{aligned}$$

2. (Wiener integrals).
We consider the Itô integral

$$I_t \;:=\; \int_0^t h(s)\, dB_s\,, \qquad 0 \leq t \leq 1\,,$$

of a *deterministic* function $h \in L^2([0,1], ds)$ w.r.t. a Brownian motion $(B_s)_{s\geq 0}$.

a) Show that I_t is normally distributed with mean zero and variance

$$\tau(t) \;=\; \int_0^t h(r)^2\,dr\,.$$

b) More generally, prove that increments of $(I_t)_{t\in[0,1]}$ over disjoint intervals are independent with law

$$I_t - I_s \;\sim\; N(0, \tau(t)-\tau(s)) \qquad \text{for any } 0 \le s \le t.$$

c) Conclude that the process $(I_t)_{t\in[0,1]}$ has the same law on $C([0,1],\mathbb{R})$ as the time-changed Brownian motion $t \mapsto B_{\tau(t)}$.

3. (Stieltjes integrals).

a) State the definition of the Lebesgue-Stieltjes integral

$$\int_0^t f(s)\,dg(s)$$

of a locally bounded measurable function $f : [0,\infty) \to \mathbb{R}$ w.r.t. a non-decreasing continuous function $g : [0,\infty) \to \mathbb{R}$.

b) The *variation* of a function $g : [0,\infty) \to \mathbb{R}$ on the interval $[0,t]$ is defined by

$$V^{(1)}(t) \;:=\; \sup_\pi \sum_{s\in\pi} |g(s'\wedge t) - g(s\wedge t)|\,,$$

where the supremum is taken over all partitions of $\mathbb{R}_+$. Show that for continuous functions g with finite variation, both $V^{(1)}$ and $V^{(1)} - g$ are non-decreasing and continuous. Use this fact to extend the definition of the Lebesgue-Stieltjes integral to continuous integrators g of finite variation.

c) Let (π_n) be a sequence of partitions of $\mathbb{R}_+$ with $\mathrm{mesh}(\pi_n) \to 0$, and let $f, g : \mathbb{R}_+ \to \mathbb{R}$ be continuous functions. Show that if g has finite variation, then the Riemann-Stieltjes integral

$$\int_0^t g(s)\,df(s) \;:=\; \lim_{n\to\infty} \sum_{s\in\pi_n} g(s)\,(f(s'\wedge t) - f(s\wedge t))$$

exists, and the integration by parts identity

$$\int_0^t f(s)\,dg(s) \;=\; f(t)g(t) \;-\; f(0)g(0) \;-\; \int_0^t g(s)\,df(s)$$

holds. In particular, $\int g\,df$ is independent of the choice of the partition sequence (π_n).