Institut für angewandte Mathematik
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UNIVERSITÄT BONN

„Introduction to Stochastic Analysis"
Problem Sheet 5

Please hand in your solutions before 12 noon on Tuesday, November 13.

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1. (Paley-Wiener Integral).
Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$ with $B_0 = 0$. For a function $h \in C^1([0,1], \mathbb{R})$, the stochastic integral of h w.r.t. B can be defined via the integration by parts identity

$$\int_0^1 h(s) dB_s \;:=\; h(1)\, B_1 - \int_0^1 h'(s)\, B_s\, ds.$$

a) Show that the random variables $\int_0^1 h(s)dB_s$ are normally distributed with mean 0 and variance $\int_0^1 h(s)^2 ds$. In particular,

$$\mathbb{E}\left[\left(\int_0^1 h(s)dB_s\right)^2\right] = \int_0^1 h(s)^2 ds.$$

b) Use this isometry to define the integral $\int_0^1 h(s)dB_s$ for an arbitrary function $h \in L^2(0,1)$.

c) How can you extend the approach in order to define $t \mapsto \int_0^t h(s)\, dB_s$ as a continuous stochastic process for $t \in [0,1]$?

2. (Riemann-Itô sums).

a) Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$ w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$, and let $(H_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$ adapted and product measurable process, which is *continuous in mean-square*, i.e., for any $t \geq 0$,

$$H_t \in L^2(\mathbb{P}) \quad \text{and} \quad \lim_{s\to t} \mathbb{E}[(H_s - H_t)^2] = 0.$$

Show that for any sequence (π_n) of partitions of $[0,t]$ with $\mathrm{mesh}(\pi_n) \to 0$,

$$\int_0^t H_s dB_s = \lim_{n\to\infty} \sum_{s\in\pi_n} H_s(B_{s'} - B_s) \quad \text{in } L^2(\mathbb{P}).$$

b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, then $H_t := f(B_t)$ is continuous in mean-square.

3. (Martingale proof of Radon-Nikodym Theorem).
Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on $(\Omega, \mathcal{A})$ such that $\mathbb{Q}$ is *absolutely continuous w.r.t.* $\mathbb{P}$, i.e., every $\mathbb{P}$-measure zero set is also a $\mathbb{Q}$-measure zero set. A *relative density* of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on a sub-σ-algebra $\mathcal{F} \subseteq \mathcal{A}$ is an $\mathcal{F}$-measurable random variable $Z : \Omega \to [0, \infty)$ such that

$$\mathbb{Q}[A] = \int_A Z \, d\mathbb{P} \qquad \text{for any } A \in \mathcal{F}.$$

The goal of the exercise is to prove that a relative density on the σ-algebra $\mathcal{A}$ exists if it is separable. Hence let $\mathcal{A} = \sigma(\bigcup \mathcal{F}_n)$ where $(\mathcal{F}_n)$ is a filtration consisting of σ-algebras $\mathcal{F}_n$ that are generated by finitely many disjoints sets $B_{n,i}$ $(i = 1, \ldots, k_n)$ such that $\bigcup_i B_{n,i} = \Omega$.

a) Write down explicitly relative densities Z_n of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on each $\mathcal{F}_n$, and show that (Z_n) is a non-negative martingale under $\mathbb{P}$.

b) Prove that the limit $Z_\infty = \lim Z_n$ exists both $\mathbb{P}$-almost surely and in $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

c) Conclude that Z_∞ is a relative density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on $\mathcal{A}$.

4. (Simulation of stochastic integrals).
Let (B_t) be a standard Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Use Riemann sum approximations to simulate the stochastic processes

$$I_t = \int_0^t B_s \, dB_s \qquad \text{and} \qquad \hat{I}_t = \int_0^t B_s \, \hat{d}B_s \qquad \text{for } t \in [0, 1].$$

Here the first integral is an Itô integral, and the second integral is a backward Itô integral.

b) Plot the graphs of samples from the difference process $\hat{I}_t - I_t$. What do you observe ? State a conjecture.

c) Can you prove your conjecture ?
Hint: Compute the expectations and variances of the Riemann sum approximations $\hat{I}_t^{(h)} - I_t^{(h)}$ *to* $\hat{I}_t - \hat{I}_t$ *for an equidistant partition of* $[0, 1]$ *with mesh size* h. *What happens as* $h \downarrow 0$ *?*