1. (Paley-Wiener Integral).
Let \((B_t)_{t \geq 0}\) be a one-dimensional Brownian motion on \((\Omega, \mathcal{A}, \mathbb{P})\) with \(B_0 = 0\). For a function \(h \in C^1([0,1], \mathbb{R})\), the stochastic integral of \(h\) w.r.t. \(B\) can be defined via the integration by parts identity
\[
\int_0^1 h(s)dB_s := h(1) B_1 - \int_0^1 h'(s) B_s ds.
\]

a) Show that the random variables \(\int_0^1 h(s)dB_s\) are normally distributed with mean 0 and variance \(\int_0^1 h(s)^2ds\). In particular,
\[
\mathbb{E}\left[\left(\int_0^1 h(s)dB_s\right)^2\right] = \int_0^1 h(s)^2ds.
\]

b) Use this isometry to define the integral \(\int_0^1 h(s)dB_s\) for an arbitrary function \(h \in L^2(0,1)\).

c) How can you extend the approach in order to define \(t \mapsto \int_0^t h(s)dB_s\) as a continuous stochastic process for \(t \in [0,1]\) ?

2. (Riemann-Itô sums).

a) Let \((B_t)_{t \geq 0}\) be a one-dimensional Brownian motion on \((\Omega, \mathcal{A}, \mathbb{P})\) w.r.t. a filtration \((\mathcal{F}_t)_{t \geq 0}\), and let \((H_t)_{t \geq 0}\) be an \((\mathcal{F}_t)_{t \geq 0}\) adapted and product measurable process, which is continuous in mean-square, i.e., for any \(t \geq 0\),
\[
H_t \in L^2(\mathbb{P}) \quad \text{and} \quad \lim_{s \to t} \mathbb{E}[(H_s - H_t)^2] = 0.
\]
Show that for any sequence \((\pi_n)\) of partitions of \([0,t]\) with mesh(\(\pi_n\)) \(\to 0\),
\[
\int_0^t H_sdB_s = \lim_{n \to \infty} \sum_{s \in \pi_n} H_s(B_s' - B_s) \quad \text{in} \ L^2(\mathbb{P}).
\]

b) Show that if \(f : \mathbb{R} \to \mathbb{R}\) is a Lipschitz continuous function, then \(H_t := f(B_t)\) is continuous in mean-square.
3. (Martingale proof of Radon-Nikodym Theorem).
Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on $(\Omega, \mathcal{A})$ such that $\mathbb{Q}$ is absolutely continuous w.r.t. $\mathbb{P}$, i.e., every $\mathbb{P}$-measure zero set is also a $\mathbb{Q}$-measure zero set. A relative density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on a sub-$\sigma$-algebra $\mathcal{F} \subseteq \mathcal{A}$ is an $\mathcal{F}$-measurable random variable $Z : \Omega \to [0, \infty)$ such that
\[ \mathbb{Q}[A] = \int_A Z \, d\mathbb{P} \quad \text{for any } A \in \mathcal{F}. \]
The goal of the exercise is to prove that a relative density on the $\sigma$-algebra $\mathcal{A}$ exists if it is separable. Hence let $\mathcal{A} = \sigma(\bigcup \mathcal{F}_n)$ where $(\mathcal{F}_n)$ is a filtration consisting of $\sigma$-algebras $\mathcal{F}_n$ that are generated by finitely many disjoints sets $B_{n,i}$ ($i = 1, \ldots, k_n$) such that $\bigcup_i B_{n,i} = \Omega$.

a) Write down explicitly relative densities $Z_n$ of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on each $\mathcal{F}_n$, and show that $(Z_n)$ is a non-negative martingale under $\mathbb{P}$.

b) Prove that the limit $Z_\infty = \lim_n Z_n$ exists both $\mathbb{P}$-almost surely and in $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

c) Conclude that $Z_\infty$ is a relative density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on $\mathcal{A}$.

4. (Simulation of stochastic integrals).
Let $(B_t)$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Use Riemann sum approximations to simulate the stochastic processes
\[ I_t = \int_0^t B_s \, dB_s \quad \text{and} \quad \hat{I}_t = \int_0^t B_s \, \hat{d}B_s \quad \text{for } t \in [0, 1]. \]

Here the first integral is an Itô integral, and the second integral is a backward Itô integral.

b) Plot the graphs of samples from the difference process $\hat{I}_t - I_t$. What do you observe? State a conjecture.

c) Can you prove your conjecture?
\textit{Hint: Compute the expectations and variances of the Riemann sum approximations $\hat{I}_t^{(h)} - I_t^{(h)}$ to $\hat{I}_t - I_t$ for an equidistant partition of $[0, 1]$ with mesh size $h$. What happens as $h \downarrow 0$?}