Institut für angewandte Mathematik
Wintersemester 2018/19
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„Introduction to Stochastic Analysis"
Problem Sheet 4

Please hand in your solutions before 12 noon on Tuesday, November 6, into the marked pigeonholes opposite to the maths library.

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1. (Uniform integrability).

a) For which sequences (a_n) of real numbers are the random variables

$$X_n \;= a_n \cdot I_{(0,1/n)}\ , \qquad n \in \mathbb{N},$$

uniformly integrable w.r.t the uniform distribution on the interval $(0,1)$?

b) Show that the exponential martingale $M_t = \exp(B_t - t/2)$ of a 1-dimensional Brownian motion is not uniformly integrable.

c) Let $(M_n)_{n\in\mathbb{Z}_+}$ be an $(\mathcal{F}_n)$ martingale with $\sup \mathbb{E}[\,|M_n|^p\,] < \infty$ for some $p > 1$. Prove that (M_n) converges almost surely and in L^1, and $M_n = \mathbb{E}[M_\infty \mid \mathcal{F}_n]$ for all $n \geq 0$. Hence, conclude that $|M_n - M_\infty|^p$ is uniformly integrable, and $M_n \to M_\infty$ in L^p.

2. (Angle bracket process and martingale convergence). Suppose that $(M_n)_{n\in\mathbb{Z}_+}$ is a square integrable $(\mathcal{F}_n)$ martingale on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with conditional variance process $\langle M\rangle_n$. We set $\langle M\rangle_\infty := \lim_{n\to\infty}\langle M\rangle_n$.

a) Let T be an $(\mathcal{F}_n)$ stopping time, and let $M_n^T := M_{n\wedge T}$ denote the stopped martingale. Show that almost surely,

$$\langle M^T\rangle_n \;=\; \langle M\rangle_{n\wedge T} \qquad \text{for all } n \geq 0.$$

b) Let $a > 0$. Show that $T_a := \inf\{n \geq 0 : \langle M\rangle_{n+1} > a\}$ is an $(\mathcal{F}_n)$ stopping time.

c) Prove that the stopped martingale $(M_n^{T_a})$ converges almost surely and in L^2.

d) Hence conclude that (M_n) converges almost surely on the set $\{\langle M\rangle_\infty < \infty\}$.

3. (Backward martingale convergence and law of large numbers). Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a *decreasing* sequence of sub-σ-algebras on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Prove that for every random variable $X \in \mathcal{L}^1(\Omega, \mathcal{A}, P)$, the limit $M_{-\infty}$ of the sequence $M_{-n} := E[X \mid \mathcal{F}_n]$ as $n \to \infty$ exists almost surely and in L^1, and

$$M_{-\infty} \;=\; \mathbb{E}[X \mid \bigcap \mathcal{F}_n] \qquad \text{almost surely.}$$

Hint: Apply Doob's upcrossing inequality to the martingales $(M_{k-n})_{k=0,1,\ldots n}$.

b) Now let (X_n) be a sequence of i.i.d. random variables in $\mathcal{L}^1(\Omega, \mathcal{A}, \mathbb{P})$, and let $\mathcal{F}_n = \sigma(S_n, S_{n+1}, \ldots)$, where $S_n = X_1 + \ldots + X_n$. Prove that almost surely,

$$\mathbb{E}[X_1 \mid \mathcal{F}_n] \;=\; \frac{S_n}{n},$$

and conclude that the strong Law of Large Numbers holds:

$$\frac{S_n}{n} \longrightarrow E[X_1] \qquad \text{almost surely.}$$

4. (Law of the iterated logarithm). Let $(B_t)_{t \geq 0}$ be a one dimensional Brownian motion with $B_0 = 0$. Recall from the lectures that almost surely,

$$\limsup_{t \downarrow 0} \frac{B_t}{h(t)} \;\leq\; +1, \qquad \text{where} \qquad h(t) = \sqrt{2t \log\log t^{-1}}.$$

Complete the proof of the Law of Iterated Logarithm, i.e., show that almost surely,

$$\limsup_{t \downarrow 0} \frac{B_t}{h(t)} \;=\; +1$$

To this end, you may proceed in the following way:

a) Show that almost surely,

$$\liminf_{t \downarrow 0} \frac{B_t}{h(t)} \;\geq\; -1.$$

b) Let $\theta \in (0,1)$ and consider the increments $Z_n = B_{\theta^n} - B_{\theta^{n+1}}, n \in \mathbb{N}$. Show that if $\epsilon > 0$, then

$$P[Z_n > (1-\epsilon)h(\theta^n) \quad \text{infinitely often}] \;=\; 1.$$

(Hint: $\int_x^\infty \exp(-z^2/2)dz \;\geq\; (x^{-1} - x^{-3}) \exp(-x^2/2)$*.)*

c) Using the statements in a) and b), conclude that

$$\limsup_{t \searrow 0} \frac{B_t}{h(t)} \;\geq\; 1 - \epsilon \qquad P\text{-almost surely for every } \epsilon > 0.$$

Hence complete the proof of the LIL.