Institut für angewandte Mathematik
Wintersemester 2018/19
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„Introduction to Stochastic Analysis"
Problem Sheet 3

Please hand in your solutions before 1 pm on Tuesday, October 30, into the marked pigeonholes opposite to the maths library.

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1. (Martingales and stopping times of Brownian motion).
Let $(B_t)_{t\geq 0}$ be a d-dimensional Brownian motion. Show that:

a) The following processes are martingales w.r.t. each of the filtrations $(\mathcal{F}_t^B)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^B$ denotes the right continuous filtration:

   i) The coordinate processes $B_t^{(i)}$, $1 \leq i \leq d$,
   ii) $B_t^{(i)} B_t^{(j)} - t \cdot \delta_{ij}$ for any $1 \leq i, j \leq d$,
   iii) $\exp(\alpha \cdot B_t - \frac{1}{2}|\alpha|^2 t)$ for any $\alpha \in \mathbb{R}^d$.

b) If *all* sample paths $t \mapsto B_t(\omega)$ are continuous, then for every closed set $A \subset \mathbb{R}^d$, the first hitting time $T_A = \inf\{t \geq 0 : B_t \in A\}$ is a stopping time w.r.t. $(\mathcal{F}_t^B)_{t\geq 0}$.

c) For an open set $U \subset \mathbb{R}^d$, T_U is a stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ but not necessarily w.r.t. $(\mathcal{F}_t^B)_{t\geq 0}$.

2. (Ruin probabilities and passage times revisited).
Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion starting at 0. For $a, b > 0$ let

$$T = \inf\{t \geq 0 \,:\, B_t \notin (-b, a)\} \qquad \text{and} \qquad T_a = \inf\{t \geq 0 \,:\, B_t = a\}$$

denote the first exit time from the interval $(-b, a)$, and the first hitting time of a, respectively. You may assume without proof that both stopping times are almost surely finite. Show that:

a) *Ruin probabilities:* $\mathbb{P}[B_T = a] = b/(a+b)$, $\quad \mathbb{P}[B_T = -b] = a/(a+b)$;

b) *Mean exit time:* $\mathbb{E}[T] = a \cdot b$, and $\mathbb{E}[T_a] = \infty$;

c) *Laplace transform of passage times:* $\mathbb{E}[\exp(-sT_a)] = \exp(-a\sqrt{2s})$ for any $s > 0$;

d) The distribution of T_a on $(0, \infty)$ is absolutely continuous with density

$$f_{T_a}(t) \;=\; a \cdot (2\pi t^3)^{-1/2} \cdot \exp(-a^2/2t).$$

3. (Stopping times). Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Let S and T be $(\mathcal{F}_t)$ stopping times. Show that the following properties hold:

   (i) $T \wedge S$, $T \vee S$ and $T + S$ are again $(\mathcal{F}_t)$ stopping times.

   (ii) $\mathcal{F}_T$ is a σ-algebra, and T is $\mathcal{F}_T$-measurable.

   (iii) $S \leq T \Rightarrow \mathcal{F}_S \subseteq \mathcal{F}_T$;

   (iv) $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$;

   (v) The events $\{S < T\}$, $\{S \leq T\}$ and $\{S = T\}$ are all contained in $\mathcal{F}_S \cap \mathcal{F}_T$.

b) Show that an integrable $(\mathcal{F}_t)$ adapted continuous process (M_t) is a martingale if and only if

$$\mathbb{E}[M_T] = \mathbb{E}[M_0] \qquad \text{for any bounded stopping time } T.$$

4. (Simulation of Ornstein-Uhlenbeck processes II). A two dimensional *Ornstein-Uhlenbeck process* is a stochastic process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}^2$ that solves a stochastic differential equation $dX_t = AX_t\,dt + \sigma\,dB_t$, $X_0 = x_0$, i.e.,

$$X_t \;=\; x_0 + \int_0^t A\,X_s\,ds \;+\; \sigma\,B_t \qquad \text{for all } t \in [0,\infty), \tag{1}$$

where $(B_t)_{t\geq 0}$ is a two dimensional Brownian motion, A is a 2×2 matrix, and $\sigma \in (0,\infty)$ and the initial value $x_0 \in \mathbb{R}^2$ are given constants..

a) Write down a time-discretization of (1), where $t \in h\mathbb{Z}_+$ for a given step size $h > 0$.

b) Simulate a sample path of a general two dimensional Ornstein Uhlenbeck process on a time interval $[0, t_{\max}]$, and plot the trajectory.

c) Run the simulation for the following choices of A and σ, $t_{\max} = 40$ and $x_0 = (1,0)$.

   (i) *Two dimensional Brownian motion:* $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma = 1$.

   (ii) *Standard two dimensional OU process:* $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma = 1$.

   (iii) *Randomly perturbed rotation:* $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma = 0.1$ and $\sigma = 1$.

   (iv) *Randomly perturbed rotation with damping:* $A = \begin{pmatrix} -\gamma & 1 \\ -1 & -\gamma \end{pmatrix}$, $\sigma, \gamma \in \{0.1, 1\}$.

   (v) *Damping in one component:* $A = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix}$, $\sigma, \gamma \in \{0.1, 1\}$.

Hint: It might make sense to choose a higher resolution and thin lines for the plots. For example in Python if the numerical solution is stored in a $2 \times$ steps *array* sde:

```
plt.figure(figsize=(7,7), dpi=500)
plt.plot(sde[0],sde[1],linewidth=.2)
plt.show()
```

Matrix multiplication in Python: np.matmul(A,B)