Institut für angewandte Mathematik
Wintersemester 2018/19
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„Introduction to Stochastic Analysis"
Problem Sheet 2

Please hand in your solutions before 12 noon on Tuesday, October 23, into the marked pigeonholes opposite to the maths library.

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1. (Bounds for random walks and bin packing).

a) Let $(S_n)_{n\geq 0}$ be a simple random walk on $\mathbb{Z}$, i.e., $S_n = U_1 + \cdots + U_n$, where the r.v.'s U_i are i.i.d. with $\mathbb{P}[U_i = 1] = p$ and $\mathbb{P}[U_i = -1] = 1 - p = q$, $p \in (0, 1/2)$. Show that the process $Z_n = (q/p)^{S_n}$ is a positive martingale, and conclude that

$$\mathbb{P}\left[\sup_{n\geq 0} S_n \geq k\right] \leq \left(\frac{p}{q}\right)^k \qquad \text{and} \qquad \mathbb{E}\left[\sup_{n\geq 0} S_n\right] \leq \frac{p}{q-p}.$$

b) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables taking values in $[0, 1]$. How many bins of size 1 are needed to pack n objects of sizes X_1, X_2 , . . ., X_n ? Let B_n be the minimal number of bins and set

$$M_k \;:=\; \mathbb{E}[B_n|\sigma(X_1, \ldots, X_k)]\,, \qquad 0 \leq k \leq n\;.$$

Show that $|M_k - M_{k-1}| \leq 1$ and conclude that

$$\mathbb{P}[|B_n - \mathbb{E}[B_n]| \geq \varepsilon] \;\leq\; 2 \cdot e^{-\frac{\varepsilon^2}{2n}}\;.$$

Remark: One can show that asymptotically, $\mathbb{E}[B_n]$ *is growing linearly in* n.

2. (Doob decomposition and local time). Let $(Y_n)_{n\geq 0}$ be a sequence of i.i.d. r.v.'s such that $\mathbb{P}[Y_k = 1] = \mathbb{P}[Y_k = -1] = 1/2$. For $n \geq 0$ we set $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$.

a) Let $S_n = Y_1 + \cdots + Y_n$. Compute the conditional variance process $\langle S\rangle_n$.

b) Now consider the process defined by

$$M_n = \sum_{k=1}^{n} \operatorname{sign}(S_{k-1})Y_k\,, \quad \text{where} \quad \operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Show that $(M_n)_{n\geq 0}$ is a square integrable martingale and compute $\langle M\rangle_n$.

c) Compute the Doob decomposition of $(|S_n|)_{n\geq 0}$. Show that M_n is measurable with respect to the σ-algebra $\sigma(|S_1|, \ldots, |S_n|)$.

3. (CRR model of stock market). In the Cox-Ross-Rubinstein binomial model of mathematical finance, the price of an asset is changing during each period either by a factor $1+a$ or by a factor $1+b$ with $a, b \in (-1, \infty)$ such that $a < b$. We can model the price evolution in a fixed number N of periods by a stochastic process

$$S_n = S_0 \cdot \prod_{i=1}^{n} X_i, \qquad n = 0, 1, 2, \dots, N,$$

defined on $\Omega = \{1+a, 1+b\}^N$, where the initial price S_0 is a given constant, and $X_i(\omega) = \omega_i$. Taking into account a constant interest rate $r > 0$, the discounted stock price after n periods is

$$\tilde{S}_n \;=\; S_n/(1+r)^n \;=\; S_0 \cdot \prod_{i=1}^{n} \frac{X_i}{1+r}.$$

A probability measure $\mathbb{P}$ on Ω is called a *martingale measure* if the discounted stock price is a martingale w.r.t. $\mathbb{P}$ and the filtration $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Suppose that $a < r < b$.

a) Prove that $\mathbb{P}$ is a martingale measure if and only if $X_1, \dots, X_N$ are independent under $\mathbb{P}$ with

$$\mathbb{P}[X_n = 1+b] \;=\; \frac{r-a}{b-a} \qquad \text{and} \qquad \mathbb{P}[X_n = 1+a] \;=\; \frac{b-r}{b-a}.$$

b) *Completeness:* Prove that for any function $F : \Omega \to \mathbb{R}$ there exists a constant V_0 and a predictable sequence $(\Phi_n)_{1 \le n \le N}$ such that $F = V_N := V_0 + (\Phi_\bullet S)_N$, or, equivalently,

$$(1+r)^{-N} F \;=\; \widetilde{V}_N \;=\; V_0 + (\Phi_\bullet \widetilde{S})_N.$$

Hence in the CRR model, any $\mathcal{F}_N$-measurable function F can be replicated by a predictable trading strategy. Market models with this property are called *complete*.

Hint: Prove inductively that for $n = N, N-1, \dots, 0$, $\tilde{F} = F/(1+r)^N$ can be represented as

$$\tilde{F} = \widetilde{V}_n + \sum_{i=n+1}^{N} \Phi_i \cdot (\widetilde{S}_i - \widetilde{S}_{i-1})$$

with an $\mathcal{F}_n$-measurable function $\widetilde{V}_n$ and a predictable sequence $(\Phi_i)_{n+1 \le i \le N}$.

c) *Option pricing:* Derive a general formula for the no-arbitrage price of an option with payoff function $F : \Omega \to \mathbb{R}$ in the CRR model. Compute the no-arbitrage price for a European call option with maturity N and strike K explicitly.

4. (Simulation of Ornstein-Uhlenbeck processes I). An *Ornstein-Uhlenbeck process* is a solution of a stochastic differential equation $dX_t = -\gamma X_t \, dt + \sigma \, dB_t$, $X_0 = x_0$, i.e.,

$$X_t \;=\; x_0 - \int_0^t \gamma \, X_s \, ds \;+\; \sigma \, B_t \qquad \text{for all } t \in [0, \infty), \tag{1}$$

where $x_0 \in \mathbb{R}$ and $\gamma, \sigma \in (0, \infty)$ are given constants, and $(B_t)_{t \ge 0}$ is a Brownian motion.

a) Write down a time-discretization of (1), where $t \in h\mathbb{Z}_+$ for a given step size $h > 0$.

b) Simulate a given number k of sample paths of an OU process on a time interval $[0, t_{\max}]$, and plot the result. Run the simulation for different values of h, x_0, γ and σ.

Remark on programming exercises: You can use a language of your choice. Model solutions will be provided in Python with Jupyter notebooks. Links to a brief introduction to Python and Jupyter notebooks and some examples can be found on the course homepage.