Institut für angewandte Mathematik
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„Introduction to Stochastic Analysis"
Problem Sheet 13

Please hand in your solutions before 12 noon on Tuesday, January 22nd.

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1. (Stochastic oscillator).

a) Let A and σ be $d \times d$–matrices, let $a \in \mathbb{R}^d$, and suppose that $(B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$.

(i) Solve the SDE

$$dZ_t \;=\; (AZ_t + a)\,dt \;+\; \sigma\,dB_t\;, \qquad Z_0 = z_0.$$

(ii) Show that Z_t is a normally distributed random vector with mean vector $m(t)$ and covariance matrix $C(t)$ where m and C are solutions of the ordinary differential equations

$$\dot{m} = Am + a, \qquad \dot{C} = AC + CA^T + \sigma\sigma^T.$$

b) Small displacements from equilibrium (e.g. of a pendulum) with stochastic reset force are described by an SDE of type

$$\begin{aligned} dX_t &= V_t\,dt \\ dV_t &= -X_t\,dt \;+\; dB_t \end{aligned}$$

with a one-dimensional Brownian motion $(B_t)_{t\geq 0}$. In complex notation:

$$dZ_t = -iZ_t\,dt \;+\; i\,dB_t, \qquad \text{where } Z_t = X_t + iV_t.$$

(i) Solve the SDE with initial conditions $X_0 = x_0$, $V_0 = v_0$.

(ii) Show that X_t is a normally distributed random variable with mean given by the solution of the corresponding deterministic equation.

2. (Random rotations). Let $(B_t)_{t\geq 0}$ be a d-dimensional Brownian motion, and suppose that $(O_t)_{t\geq 0}$ is a continuous adapted process taking values in the orthogonal $d\times d$ matrices. Prove that the process

$$X_t \;=\; \int_0^t O_s\,dB_s$$

is again a d-dimensional Brownian motion.

3. (Cox-Ingersoll-Ross model).
Let $(B_t)_{t\geq 0}$ be a Brownian motion. The Cox-Ingersoll-Ross model aims to describe, for example, an interest rate process $(R_t)_{t\geq 0}$ or a stochastic volatility process and is given by

$$dR_t \;=\; (\alpha - \beta R_t)dt + \sigma\sqrt{R_t}dB_t, \qquad R_0 = x_0 > 0,$$

where $\alpha, \beta, \sigma > 0$. It can be shown that the SDE admits a strong solution.

a) Compute the corresponding *scale function* and study the asymptotic behaviour of R_t depending on the parameters α, β and σ.

b) Suppose that $2\alpha \geq \sigma^2$. We study further properties of R_t :

(i) By applying Itô's formula, show that $E[|R_t|^p] < \infty$ for any $t > 0$ and $p \geq 1$.

(ii) Compute the expectation of R_t. *(Hint: Apply Itô's formula to $f(t,x) = e^{\beta t}x$.)*

(iii) Proceed in a similar way to compute $\mathrm{Var}[R_t]$, and determine $\lim\limits_{t\to\infty} \mathrm{Var}[R_t]$.

4. (Black-Scholes model).
A stock price is modeled by a geometric Brownian Motion $(S_t)_{t\geq 0}$ with parameters $\alpha, \sigma > 0$. We assume that the interest rate is equal to a real constant r for all times. Let $c(t,x)$ be the value of an option at time t if the stock price at that time is $S_t = x$. Suppose that $c(t, S_t)$ is replicated by a hedging portfolio, i.e., there is a trading strategy holding ϕ_t shares of stock at time t and putting the remaining portfolio value $V_t - \phi_t S_t$ in the money market account with fixed interest rate r so that the total portfolio value V_t at each time t agrees with $c(t, S_t)$.
"Derive" the *Black-Scholes partial differential equation*

$$\frac{\partial c}{\partial t}(t,x) + rx\frac{\partial c}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c}{\partial x^2}(t,x) \;=\; rc(t,x) \tag{1}$$

and the *delta-hedging rule*

$$\phi_t \;=\; \frac{\partial c}{\partial x}(t, S_t) \qquad (=: \text{ Delta}). \tag{2}$$

(Hint: Consider the discounted portfolio value $\tilde{V}_t = e^{-rt}V_t$ and, correspondingly, $e^{-rt}c(t,S_t)$. Compute the Ito differentials, and conclude that both processes coincide if c is a solution to (1) and ϕ_t is given by (2).)