Institut für angewandte Mathematik
Wintersemester 2018/19
Andreas Eberle, Alessandra Occelli

„Introduction to Stochastic Analysis"
Problem Sheet 12

Please hand in your solutions before 12 noon on Tuesday, January 15th.

---

1. (Feynman and Kac at the stock exchange). The price of a security is modeled by geometric Brownian motion (X_t) with parameters $\alpha, \sigma > 0$. At a price x we have a cost $V(x)$ per unit of time. The total cost up to time t is then given by

$$A_t = \int_0^t V(X_s)ds .$$

Suppose that u is a bounded solution to the PDE

$$\frac{\partial u}{\partial t} = \mathcal{L}u - \beta V u , \qquad \text{where} \qquad \mathcal{L} = \frac{\sigma^2}{2} x^2 \frac{d^2}{dx^2} + \alpha x \frac{d}{dx} .$$

Show that the Laplace transform of A_t is given by $E_x\left[e^{-\beta A_t}\right] = u(t, x)$.

2. (Variation of constants II).
We consider nonlinear stochastic differential equations

$$dX_t = f(t, X_t)\, dt + c(t) X_t\, dB_t, \qquad X_0 = x,$$

where $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R}^+ \to \mathbb{R}$ are continuous (deterministic) functions.

a) Find an explicit solution Z_t of the equation with $f \equiv 0$.

b) To solve the equation in the general case, use the Ansatz $X_t = C_t \cdot Z_t$. Show that the SDE gets the form

$$\frac{dC_t(\omega)}{dt} = f(t, Z_t(\omega) \cdot C_t(\omega))/Z_t(\omega) , \qquad C_0 = x. \tag{1}$$

Note that for each $\omega \in \Omega$, this is a *deterministic* differential equation for the function $t \mapsto C_t(\omega)$. We can therefore solve (1) with ω as a parameter to find $C_t(\omega)$.

c) Apply this method to solve the stochastic differential equation

$$dX_t = \frac{1}{X_t} dt + \alpha X_t\, dB_t , \qquad X_0 = x > 0 , \qquad \alpha \in \mathbb{R}.$$

d) Apply the method to study the solution of the stochastic differential equation

$$dX_t = X_t^\gamma \, dt + \alpha X_t \, dB_t \; , \qquad X_0 = x > 0 \; ,$$

where α and γ are constants. For which values of γ do we get explosion?

3. (Lévy Area).
If $c(t) = (x(t), y(t))$ is a smooth curve in $\mathbb{R}^2$ with $c(0) = 0$, then

$$A(t) \;=\; \int_0^t (x(s)y'(s) - y(s)x'(s))\, ds \;=\; \int_0^t x\, dy - \int_0^t y\, dx$$

describes the area that is covered by the secant from the origin to $c(s)$ in the interval $[0, t]$. Analogously, for a two-dimensional Brownian motion $B_t = (X_t, Y_t)$ with $B_0 = 0$, one defines the *Lévy Area*

$$A_t \;:=\; \int_0^t X_s \, dY_s \;-\; \int_0^t Y_s \, dX_s \, .$$

a) Let $\alpha(t)$, $\beta(t)$ be C^1-functions, $p \in \mathbb{R}$, and

$$V_t \;=\; ipA_t \;-\; \frac{\alpha(t)}{2} \left(X_t^2 + Y_t^2\right) \;+\; \beta(t) \, .$$

Show that e^{V_t} is a local martingale provided $\alpha'(t) = \alpha(t)^2 - p^2$ and $\beta'(t) = \alpha(t)$.

b) Let $t_0 \in [0, \infty)$. Show that the solutions of the ordinary differential equations for α and β with $\alpha(t_0) = \beta(t_0) = 0$ are

$$\begin{aligned} \alpha(t) =& p \cdot \tanh(p \cdot (t_0 - t)) \, , \\ \beta(t) =& -\log\cosh(p \cdot (t_0 - t)) \, . \end{aligned}$$

Hence conclude that

$$E\left[e^{ipA_{t_0}}\right] \;=\; \frac{1}{\cosh(pt_0)} \qquad \forall\, p \in \mathbb{R} \, .$$

c) Show that the distribution of A_t is absolutely continuous with density

$$f_{A_t}(x) \;=\; \frac{1}{2t \cosh(\frac{\pi x}{2t})} \, .$$