Institut für angewandte Mathematik
Wintersemester 2018/19
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„Introduction to Stochastic Analysis"
Problem Sheet 11

Please hand in your solutions before 12 noon on Tuesday, January 8th, 2019.

We wish you a merry christmas and a happy new year 2019!

1. (Quadratic variation of Itô integrals). Suppose that $X : [0, \infty) \to \mathbb{R}$ is a continuous function with continuous quadratic variation $[X]$ w.r.t. a fixed sequence (π_n) of partitions s.t. $\mathrm{mesh}(\pi_n) \to 0$.

a) Let $F \in C^1(\mathbb{R})$. Show that the quadratic variation of $t \mapsto F(X_t)$ along (π_n) is given by

$$[F(X)]_t \;=\; \int_0^t F'(X_s)^2 \, d[X]_s.$$

b) Conclude that for $f \in C^1(\mathbb{R})$, the Itô integral $I_t = \int_0^t f(X_s)\, dX_s$ has quadratic variation

$$[I(f)]_t \;=\; \int_0^t f(X_s)^2 \, d[X]_s \,.$$

2. (Complex-valued Brownian motion). A complex-valued Brownian motion is given by $B_t = B_t^1 + iB_t^2$ with independent one-dimensional Brownian motions B^1 and B^2.

a) Prove that for any holomorphic function F,

$$F(B_t) \;=\; F(B_0) + \int_0^t F'(B_s)\, dB_s \,,$$

where F' denotes the complex derivative of F.
Hint: Use the Cauchy-Riemann equations.

b) Solve the complex-valued SDE $\quad dZ_t \;=\; \alpha\, Z_t\, dB_t, \quad \alpha \in \mathbb{C}\,.$

3. (Heat equation on an interval). Let $V : (a,b) \to [0,\infty)$ be continuous and bounded, and suppose that $u \in C^{1,2}([0,\infty)\times(a,b))$ $(-\infty < a < b < \infty)$ is an up to the boundary continuous and bounded solution of the heat equation

$$\frac{\partial u}{\partial t}(t,x) \;=\; \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) \;-\; V(x)\,u(t,x)$$

with initial and boundary conditions

$$u(0,x) = f(x), \quad u(t,a) = h(t), \quad u(t,b) = k(t).$$

By considering an appropriate martingale show that

$$\begin{aligned} u(t,x) \;&=\; \mathbb{E}_x\left[f(B_t)\,\exp\left(-\int_0^t V(B_s)\,ds\right)\;;\; t \le T_a \wedge T_b\right] \\ &\quad + \mathbb{E}_x\left[h(t-T_a)\,\exp\left(-\int_0^{T_a} V(B_s)\,ds\right)\;;\; T_a < t \wedge T_b\right] \\ &\quad + \mathbb{E}_x\left[k(t-T_b)\,\exp\left(-\int_0^{T_b} V(B_s)\,ds\right)\;;\; T_b < t \wedge T_a\right]. \end{aligned}$$

4. (Local time of Brownian motion). What happens if we try to apply Itô's formula to $g(B_t)$ when $(B_t)_{t\ge 0}$ is a one dimensional Brownian motion and $g(x) = |x|$?
Since g is not differentiable at 0, we consider the smooth approximations

$$g_\epsilon(x) \;:=\; \begin{cases} |x| & \text{if } |x| \ge \epsilon, \\ \frac{1}{2}(\epsilon + \frac{x^2}{\epsilon}) & \text{if } |x| < \epsilon, \end{cases} \qquad \text{where } \epsilon > 0.$$

a) Show that almost surely,

$$g_\epsilon(B_t) \;=\; g_\epsilon(B_0) \;+\; \int_0^t g_\epsilon'(B_s)\,dB_s \;+\; \frac{1}{2\epsilon}\,\lambda\left[\{s \in [0,t] : B_s \in (-\epsilon,\epsilon)\}\right].$$

b) Prove that as $\epsilon \to 0$,

$$\int_0^t g_\epsilon'(B_s)\,\mathbb{1}_{(-\epsilon,\epsilon)}(B_s)\,dB_s \;\to\; 0$$

in an appropriate sense to be specified.

c) Conclude that almost surely,

$$|B_t| \;=\; |B_0| \;+\; \int_0^t \mathrm{sign}(B_s)\,dB_s \;+\; L_t, \tag{1}$$

where we set $\mathrm{sign}(x) := -1$ for $x \le 0$ (!) and $\mathrm{sign}(x) := +1$ for $x > 0$, and

$$L_t \;=\; \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\,\lambda\left[\{s \in [0,t] : B_s \in (-\epsilon,\epsilon)\}\right].$$

The process L_t is called the *local time* for Brownian motion at 0.