Institut für angewandte Mathematik
Wintersemester 2018/19
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„Introduction to Stochastic Analysis"
Problem Sheet 10

Please hand in your solutions before 12 noon on Tuesday, December 18.

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1. (Variation of constants). Solve the stochastic differential equation

$$dY_t = r\,dt + \alpha\,Y_t\,dB_t$$

for a one-dimensional Brownian motion (B_t) and constants $r, \alpha \in \mathbb{R}$.
Hint: Consider first the case $r = 0$, then try a variation of constants ansatz.

2. (Recurrence and transience of Brownian motion).

a) Show that for a Brownian motion (B_t) in $\mathbb{R}^2$, every non-empty open ball D is recurrent, i.e.,

$$\mathbb{P}[\forall\, t \geq 0\ \exists\, s \geq t : B_s \in D] \;=\; 1.$$

b) Conclude that a typical Brownian trajectory is dense in $\mathbb{R}^2$.

c) Conversely, show that for $d \geq 3$, every ball in $\mathbb{R}^d$ is transient for Brownian motion, i.e., $|B_t| \to \infty$ almost surely as $t \to \infty$.

3. (Itô diffusions). Let $(X_t, \mathbb{P}_x)$ be a solution of the SDE

$$dX_t \;=\; b(X_t)\,dt \;+\; dB_t\,, \qquad X_0 = x\ \ \mathbb{P}_x\text{–a.s.},$$

where (B_t) is a one-dimensional Brownian motion and $b \in C_b(\mathbb{R})$. Prove that:

a) Under $\mathbb{P}_x$, (X_t) solves the *time-dependent martingale problem* for the generator $\mathcal{L} \;=\; \frac{1}{2}\frac{d^2}{dx^2} \;+\; b(x)\frac{d}{dx}$, i.e., for every $u \in C^2(\mathbb{R})$,

$$M_t \;:=\; u(t, X_t) - u(0, X_0) - \int_0^t \left(\frac{\partial u}{\partial s} + \mathcal{L}u\right)(s, X_s)\,ds$$

is a local martingale.

b) *Kolmogorov's forward equation* holds, i.e.,

$$\mathbb{E}_x[f(X_t)] \;=\; f(x) \;+\; \int_0^t \mathbb{E}_x[\mathcal{L}f(X_s)]\,ds \qquad \forall\ f \in C_b^2(\mathbb{R}).$$

c) $\mu_t[A] := \mathbb{P}_x[X_t \in A]$ is a solution of $\frac{\partial \mu_t}{\partial t} = \mathcal{L}^* \mu_t$ in the distributional sense, i.e.,

$$\frac{\partial}{\partial t} \int f \, d\mu_t = \int \mathcal{L} f \, d\mu_t \qquad \forall\, f \in C_b^2(\mathbb{R}) \, .$$

d) The function $u(t,x) = \mathbb{E}_x[f(X_t)]$, $f \in C_b(\mathbb{R})$, is the unique bounded solution of

$$\frac{\partial u}{\partial t} = \mathcal{L} u \, , \qquad u(0,x) = f(x) \, .$$

(You may assume without proof the existence of a solution $u \in C_b^2$, so you only have to prove that such a solution has the stochastic representation stated above)