1. (Variation of constants). Solve the stochastic differential equation

\[ dY_t = r \, dt + \alpha Y_t \, dB_t \]

for a one-dimensional Brownian motion \((B_t)\) and constants \(r, \alpha \in \mathbb{R}\).

Hint: Consider first the case \(r = 0\), then try a variation of constants ansatz.

2. (Recurrence and transience of Brownian motion).

a) Show that for a Brownian motion \((B_t)\) in \(\mathbb{R}^2\), every non-empty open ball \(D\) is recurrent, i.e.,

\[ \mathbb{P}[\forall t \geq 0 \exists s \geq t : B_s \in D] = 1. \]

b) Conclude that a typical Brownian trajectory is dense in \(\mathbb{R}^2\).

c) Conversely, show that for \(d \geq 3\), every ball in \(\mathbb{R}^d\) is transient for Brownian motion, i.e., \(|B_t| \to \infty\) almost surely as \(t \to \infty\).

3. (Itô diffusions). Let \((X_t, \mathbb{P}_x)\) be a solution of the SDE

\[ dX_t = b(X_t) \, dt + dB_t, \quad X_0 = x \quad \mathbb{P}_x-\text{a.s.}, \]

where \((B_t)\) is a one-dimensional Brownian motion and \(b \in C_b(\mathbb{R})\). Prove that:

a) Under \(\mathbb{P}_x\), \((X_t)\) solves the \textit{time-dependent martingale problem} for the generator

\[ \mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2} + b(x) \frac{d}{dx}, \]

i.e., for every \(u \in C^2(\mathbb{R})\),

\[ M_t := u(t, X_t) - u(0, X_0) - \int_0^t \left( \frac{\partial u}{\partial s} + \mathcal{L}u \right) (s, X_s) \, ds \]

is a local martingale.

b) \textit{Kolmogorov’s forward equation} holds, i.e.,

\[ \mathbb{E}_x[f(X_t)] = f(x) + \int_0^t \mathbb{E}_x[\mathcal{L}f(X_s)] \, ds \quad \forall f \in C^2_b(\mathbb{R}). \]
c) $\mu_t[A] := \mathbb{P}_x[X_t \in A]$ is a solution of $\frac{\partial \mu}{\partial t} = \mathcal{L}^* \mu_t$ in the distributional sense, i.e.,

$$\frac{\partial}{\partial t} \int f \, d\mu_t = \int \mathcal{L} f \, d\mu_t \quad \forall f \in C^2_b(\mathbb{R}).$$

d) The function $u(t, x) = \mathbb{E}_x[f(X_t)], f \in C_b(\mathbb{R})$, is the unique bounded solution of

$$\frac{\partial u}{\partial t} = \mathcal{L} u, \quad u(0, x) = f(x).$$

(You may assume without proof the existence of a solution $u \in C^2_b$, so you only have to prove that such a solution has the stochastic representation stated above)