Institut für angewandte Mathematik
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„Introduction to Stochastic Analysis"
Problem Sheet 0

The exercises on this page will be discussed in the tutorials during the first week (i.e., between wednesday and monday). You are encouraged to study them in advance but you do not have to submit written solutions.

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1. (Revision of conditional expectations 1). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{F} \subset \mathcal{A}$ a σ-algebra, and $X : \Omega \to \mathbb{R}_+$ a non-negative random variable.

a) Define the conditional expectation $\mathbb{E}[X|\mathcal{F}]$.

b) Suppose that there exists a decomposition of Ω into disjoint sets $A_1, \dots A_n$ such that $\mathcal{F} = \sigma(\{A_1, \dots, A_n\})$. Show that

$$\mathbb{E}[X|\mathcal{F}] \;=\; \sum\nolimits_{i:\mathbb{P}[A_i]>0} \mathbb{E}[X|A_i]\, 1_{A_i}$$

is a version of the conditional expectation of X given $\mathcal{F}$.

2. (Revision of conditional expectations 2). Let $X, Y : \Omega \to \mathbb{R}_+$ be non-negative random variables. Show that $\mathbb{P}$-almost surely, the following identities hold:

a) For $\lambda \in \mathbb{R}$ we have $\mathbb{E}[\lambda X + Y|\mathcal{F}] = \lambda\mathbb{E}[X|\mathcal{F}] + \mathbb{E}[Y|\mathcal{F}]$.

b) $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]] = \mathbb{E}[X]$ and $|\mathbb{E}[X|\mathcal{F}]| \leq \mathbb{E}[|X|\,|\mathcal{F}]$.

c) If $\sigma(X)$ is independent of $\mathcal{F}$, then $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$.

d) Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be measurable spaces. If $Y : \Omega \to S$ is $\mathcal{F}$-measurable, $X : \Omega \to T$ is independent of $\mathcal{F}$ and $f : S \times T \to [0, \infty)$ is product-measurable, then

$$\mathbb{E}[f(Y, X)|\mathcal{F}](\omega) = \mathbb{E}[f(Y(\omega), X)] \qquad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega.$$

3. (Revision of conditional expectations 3). Let X, Y, Z be random variables on a joint probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We define

$$\mathbb{E}[X|Y] := \mathbb{E}[X|\sigma(Y)].$$

Show the following statements:

a) If $X, Y \in \mathcal{L}^1$ are independent and identically distributed, then $\mathbb{P}$-almost surely,

$$\mathbb{E}[X|X + Y] = \frac{1}{2}(X + Y).$$

b) If Z is independent of the pair (X, Y), then $\mathbb{P}$-almost surely,

$$\mathbb{E}[X|Y, Z] = \mathbb{E}[X|Y].$$

Is this statement still true if we only assume that X and Z are independent?

,,Introduction to Stochastic Analysis" Sheet 1

Please hand in your solutions before 1 pm on Tuesday, October 16, into the marked pigeonholes opposite to the maths library.

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1. (Conditional expectations). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{G} \subseteq \mathcal{A}$ a σ-algebra, and let X, Y, Z be random variables on $(\Omega, \mathcal{A}, \mathbb{P})$.

a) For $A \in \mathcal{A}$, consider the event $B = \{\mathbb{P}[A|\mathcal{G}] = 0\}$. Show that $B \subset A^c$ a.s.

b) Suppose that the random variables (X, Z) and (Y, Z) have the same law (in particular X and Y have common law μ). Show that, if f is a non-negative function then

$$\mathbb{E}[f(X)|Z] \;= \mathbb{E}[f(Y)|Z] \;\text{ a.s.}$$

c) Let $g : \mathbb{R} \to \mathbb{R}_+$ be measurable. Suppose that $\mathbb{P}$-almost surely, $\mathbb{E}[g(Z)|X] = h_1(X)$ and $\mathbb{E}[g(Z)|Y] = h_2(X)$. Show that $h_1 = h_2$ μ-a.s.

2. (Gaussian martingales). A process $(M_n)_{n=0,1,2,\dots}$ is called Gaussian if for every n, the vector $(M_0, \dots, M_n)$ is normally distributed. Let (M_n) be a Gaussian martingale.

a) Show that (M_n) has independent increments, i.e., the random variable $M_{n+1} - M_n$ is independent of the σ-algebra $\mathcal{F}_n = \sigma(M_0, \dots, M_n)$.

b) We set $\sigma_0^2 = \mathrm{Var}(M_0)$ and $\sigma_k^2 = \mathrm{Var}(M_k - M_{k-1})$ for $k \geq 1$. Compute the predictable increasing process $\langle M \rangle_n$ in the Doob decomposition of the submartingale (M_n^2).

c) Show that for every $\theta \in \mathbb{R}$, $Z_n^\theta = e^{\theta M_n - \frac{1}{2}\theta^2 \langle M \rangle_n}$ is a martingale. Does it converge a.s.?

3. (Martingales, supermartingales and stopping times).

a) Let (X_n) be a supermartingale s.t. $\mathbb{E}[X_n]$ is constant. Show that (X_n) is a martingale.

b) Let (X_n) be an integrable process adapted to the filtration $(\mathcal{F}_n)$. Show that (X_n) is a martingale if and only if $\mathbb{E}[X_T] = \mathbb{E}[X_0]$ for every bounded $(\mathcal{F}_n)$ stopping time T.

4. (Noisy observations). Let $X \sim \mathcal{N}(0, \sigma^2)$ and $Z_n \sim \mathcal{N}(0, c_n^2)$ $(n \in \mathbb{N})$ be independent random variables with standard deviations $\sigma, c_n > 0$. Suppose that $Y_n = X + Z_n$, and let

$$\hat{X}_n = \mathbb{E}[X|\mathcal{G}_n] \quad \text{where} \quad \mathcal{G}_n = \sigma(Y_1, \dots, Y_n).$$

a) Determine the law of the vector $(X, Y_1, \dots, Y_n)$.

b) Show that for every $y \in \mathbb{R}^n$, the conditional law of X given $(Y_1, \dots, Y_n) = y$ is Gaussian, and compute its mean and variance.

c) Compute $\hat{X}_n$ and $\mathbb{E}[(X - \hat{X}_n)^2]$.

d) Show that $\hat{X}_n \to X$ in L^2 as $n \to \infty$ if and only if $\sum_{n=1}^\infty c_n^{-2} = \infty$.

Hint: Show first that for any $a \in \mathbb{R}^d$ and any positive definite (symmetric) $d \times d$ matrix C,

$$(C + \sigma^2 a\, a^T)^{-1} \;=\; C^{-1} - \frac{C^{-1} a\, a^T C^{-1}}{\sigma^{-2} + \langle C^{-1} a, a \rangle}$$