Institut für angewandte Mathematik
Winter Semester 15/16
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Introduction to Stochastic Analysis, Problem sheet 9

Please hand in your solutions before Tuesday 12.1., 12 am.

We wish you a merry christmas and a happy new year !

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1. (Solutions of SDE). Let (B_t) be a one-dimensional Brownian motion with $B_0 = 0$. Show that the following processes solve the corresponding stochastic differential equations:

a) $X_t = B_t/(1+t)$ solves

$$dX_t = -(1+t)^{-1} X_t \, dt + (1+t)^{-1} \, dB_t \, , \qquad X_0 = 0 \, .$$

b) $X_t = \sin(B_t)$ solves

$$dX_t = -\frac{1}{2} X_t \, dt + \sqrt{1 - X_t^2} \, dB_t \, , \qquad X_0 = 0 \, ,$$

for $t < \inf \{s > 0 : B_s \notin [-\pi/2, \pi/2]\}$.

c) $(X_t, Y_t) = (t, e^t B_t)$ solves

$$\begin{bmatrix} dX_t \\ dY_t \end{bmatrix} = \begin{bmatrix} 1 \\ Y_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_t} \end{bmatrix} dB_t \, , \qquad \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \, .$$

d) $(X_t, Y_t) = (\cosh(B_t), \sinh(B_t))$ solves

$$\begin{bmatrix} dX_t \\ dY_t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X_t \\ Y_t \end{bmatrix} dt + \begin{bmatrix} Y_t \\ X_t \end{bmatrix} dB_t \, , \qquad \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \, .$$

e) For $x \geq 0$, $X_t = (x^{1/3} + \frac{1}{3} B_t)^3$ solves

$$dX_t = \frac{1}{3} X_t^{1/3} \, dt + X_t^{2/3} \, dB_t \, , \qquad X_0 = x \, .$$

Is the solution of this SDE unique ?

2. (Martingales of Brownian motion). Let (B_t) be a one-dimensional Brownian motion with $B_0 = 0$. Show that the following processes are martingales:

a) $X_t = e^{\frac{1}{2}t} \cos(B_t)$, b) $X_t = e^{\frac{1}{2}t} \sin(B_t)$, c) $X_t = (B_t + t) \exp(-B_t - \frac{1}{2}t)$.

3. (Variation of constants). Solve the stochastic differential equation

$$dY_t = r \, dt + \alpha \, Y_t \, dB_t$$

for a one-dimensional Brownian motion (B_t) and constants $r, \alpha \in \mathbb{R}$.
Hint: Consider first the case $r = 0$, then try a variation of constants ansatz.

4. (An extension of Itô's formula). Suppose that $X : [0, \infty) \to \mathbb{R}$ is a continuous function with continuous quadratic variation $[X]$ w.r.t. a sequence (π_n) of partitions s.t. $\mathrm{mesh}(\pi_n) \to 0$. Show that for $g \in C^1$ and $F \in C^2$, the Itô–Integral $\int_0^t g(X_s)\, dF(X)_s$ exists, and

$$\int_0^t g(X_s)\, dF(X)_s \;=\; \int_0^t g(X_s)\, F'(X_s)\, dX_s \;+\; \frac{1}{2}\int_0^t g(X_s)\, F''(X_s)\, d[X]_s.$$

This justifies the differential notation

$$dF(X) \;=\; F'(X)\, dX \;+\; \frac{1}{2} F''(X)\, d[X]$$

for Itô's formula, since now we are allowed to multiply this equation by $g(X)$.

5. (Poisson equation). Let $D \subset \mathbb{R}^d$ be a bounded domain, $g \in C(\bar{D})$ and $f \in C(\partial D)$. Prove by Itô's formula, that if $u \in C(\bar{D}) \cap C^2(D)$ is a solution to the Poisson equation

$$\frac{1}{2}\Delta u \;=\; -g \quad \text{on } D\,, \qquad u = f \quad \text{on } \partial D\,, \qquad\qquad \text{then}$$

$$u(x) \;=\; E_x\left[\int_0^{T_{D^c}} g(B_t)\, dt\right] \;+\; E_x\left[f(B_{T_{D^c}})\right] \qquad \forall\, x \in D\,,$$

where (B_t, P_x) is a Brownian motion with $B_0 = x$ P_x-a.s., and $T_{D^C} = \inf\{t \geq 0 : B_t \in D^C\}$.

6. (Itô diffusions). For $b \in C_b(\mathbb{R})$ let (X_t, P_x) be a solution of the SDE

$$dX_t \;=\; b(X_t)\, dt \;+\; dB_t\;, \qquad X_0 = x \;\; P_x\text{–a.s.},$$

where (B_t) is a one-dimensional Brownian motion. Prove that:

a) X_t solves under P_x the *time-dependent martingale problem* for the generator $\mathcal{L} \;=\; \frac{1}{2}\frac{d^2}{dx^2} \;+\; b(x) \cdot \frac{d}{dx}$, i.e.,

$$M_t \;:=\; u(t, X_t) - u(0, X_0) - \int_0^t \left(\frac{\partial u}{\partial s} + \mathcal{L}u\right)(s, X_s)\, ds$$

is a local martingale for any $u \in C^2$.

b) *Kolmogorov's forward equation* holds, i.e.,

$$E_x[f(X_t)] \;=\; f(x) \;+\; \int_0^t E_x[\mathcal{L}f(X_s)]\, ds \qquad \forall f \in C_b^2(\mathbb{R}).$$

c) $\mu_t(A) := P_x[X_t \in A]$ is a solution of $\frac{\partial \mu_t}{\partial t} \;=\; \mathcal{L}^* \mu_t$ in the distributional sense, i.e.,

$$\frac{\partial}{\partial t}\int f\, d\mu_t \;=\; \int \mathcal{L}f\, d\mu_t \qquad \forall\, f \in C_b^2(\mathbb{R})\,.$$

d) The function $u(t, x) = E_x[f(X_t)]$, $f \in C_b(\mathbb{R})$, is the unique bounded solution of

$$\frac{\partial u}{\partial t} \;=\; \mathcal{L}u\;, \qquad u(0, x) = f(x)\,.$$

(You may assume without proof the existence of a solution $u \in C_b^2$, so you only have to prove that such a solution has the stochastic representation stated above)