Institut für angewandte Mathematik
Winter Semester 15/16
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Introduction to Stochastic Analysis, Problem sheet 7

Please hand in your solutions with your names and the name of your tutor before Tuesday 8.12., 12 am, at the post-boxes opposite to the maths library.

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1. (Itô and Stratonovich integrals). Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{A}, P)$, and let $\pi_n = \{k2^{-n} : k \in \mathbb{Z}_+\}$ be the n-th dyadic partition of $\mathbb{R}_+$.

a) Show that for any $t \geq 0$,

$$\lim_{n\to\infty} \sum_{s\in\pi_n} (B_{s'\wedge t} - B_{s\wedge t})^2 \;=\; t \qquad \text{in } L^2(P)\,.$$

b) Starting from the definition of the Itô integral, prove that

$$\int_0^t B_s\, dB_s \;=\; \frac{1}{2}B_t^2 - \frac{1}{2}t\,.$$

c) The Stratonovich integral of (B_s) w.r.t. (B_s) is defined by

$$\int_0^t B_s \circ dB_s \;:=\; \lim_{n\to\infty} \sum_{s\in\pi_n} \frac{1}{2}(B_{s'\wedge t} + B_s)\cdot(B_{s'\wedge t} - B_s) \qquad \text{in } L^2(P)\,.$$

Show that

$$\int_0^t B_s \circ dB_s \;=\; \frac{1}{2}B_t^2\,.$$

2. (Wiener integrals). We consider the Itô integral

$$I_t \;:=\; \int_0^t h(s)\, dB_s\,, \qquad 0 \leq t \leq 1\,,$$

of a *deterministic* function $h \in L^2([0,1], ds)$ w.r.t. a Brownian motion (B_s).

a) Show that I_t is normally distributed with mean zero and variance

$$\tau(t) \;=\; \int_0^t h(r)^2\, dr\,.$$

b) More generally, prove that increments of (I_t) over disjoint intervals are independent with law

$$I_t - I_s \;\sim\; N(0, \tau(t) - \tau(s)) \qquad \text{for any } 0 \leq s \leq t.$$

c) Conclude that the process $(I_t)_{t\in[0,1]}$ has the same law on $C([0,1], \mathbb{R})$ as the time-changed Brownian motion $t \mapsto B_{\tau(t)}$.

3. (Stieltjes integrals).

a) State the definition of the Lebesgue-Stieltjes integral

$$\int_0^t f(s)\,dg(s)$$

of a locally bounded measurable function $f : [0,\infty) \to \mathbb{R}$ w.r.t. a non-decreasing continuous function $g : [0,\infty) \to \mathbb{R}$.

b) The *variation* of a function $g : [0,\infty) \to \mathbb{R}$ on the interval $[0,t]$ is defined by

$$V^{(1)}(t) \;:=\; \sup_{\pi} \sum_{s\in\pi} |g(s' \wedge t) - g(s \wedge t)|\,,$$

where the supremum is taken over all partitions of $\mathbb{R}_+$. Show that for continuous functions g with finite variation, both $V^{(1)}$ and $V^{(1)} - g$ are non-decreasing and continuous. Use this fact to extend the definition of the Lebesgue-Stieltjes integral to continuous integrators g of finite variation.

c) Let (π_n) be a sequence of partitions of $\mathbb{R}_+$ with $\mathrm{mesh}(\pi_n) \to 0$, and let $f, g : \mathbb{R}_+ \to \mathbb{R}$ be continuous functions. Show that if g has finite variation, then the Riemann-Stieltjes integral

$$\int_0^t g(s)\,df(s) \;:=\; \lim_{n\to\infty} \sum_{s\in\pi_n} g(s)\,(f(s' \wedge t) - f(s \wedge t))$$

exists, and the integration by parts identity

$$\int_0^t f(s)\,dg(s) \;=\; f(t)g(t) \;-\; f(0)g(0) \;-\; \int_0^t g(s)\,df(s)$$

holds. In particular, $\int g\,df$ is independent of the choice of the partition sequence (π_n).