Institut für angewandte Mathematik
Winter Semester 15/16
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Introduction to Stochastic Analysis, Problem sheet 6

Please hand in your solutions with your names and the name of your tutor before Tuesday 1.12., 12 am, at the post-boxes opposite to the maths library.

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1. (Paley-Wiener Integral). Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$ with $B_0 = 0$. For a function $h \in C^1([0,1], \mathbb{R})$, the stochastic integral of h w.r.t. B can be defined via the integration by parts identity

$$\int_0^1 h(s) dB_s \ := \ h(1)\, B_1 - \int_0^1 h'(s)\, B_s\, ds.$$

a) Show that the random variables $\int_0^1 h(s) dB_s$ are normally distributed with mean 0 and variance $\int_0^1 h(s)^2 ds$. In particular,

$$E\left[\left(\int_0^1 h(s) dB_s\right)^2\right] = \int_0^1 h(s)^2 ds.$$

b) Use this isometry to define the integral $\int_0^1 h(s) dB_s$ for an arbitrary function $h \in L^2(0,1)$.

c) How can you extend the approach in order to define $t \mapsto \int_0^t h(s)\, dB(s)$ as a continuous stochastic process for $t \in [0,1]$?

2. (Riemann-Itô sums).

a) Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$ w.r.t. a filtration $(\mathcal{F}_t)$, and let $(H_t)_{t\geq 0}$ be an $(\mathcal{F}_t)$ adapted and product measurable process, which is *continuous in mean-square*, i.e., for any $t \geq 0$,

$$H_t \in L^2(P) \quad \text{and} \quad \lim_{s\to t} E[(H_s - H_t)^2] = 0.$$

Show that for any sequence (π_n) of partitions of $[0,t]$ with $\mathrm{mesh}(\pi_n) \to 0$,

$$\int_0^t H_s dB_s = \lim_{n\to\infty} \sum_{s\in\pi_n} H_s (B_{s'} - B_s) \quad \text{in } L^2(P).$$

b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function then $H_t := f(B_t)$ is continuous in mean-square.

3. (Lebesgue decomposition, Lebesgue densities). Let P and Q be arbitrary (not necessarily absolutely continuous) probability measures on $(\Omega, \mathcal{A})$. A *Lebesgue density* of Q w.r.t. P is a random variable $Z : \Omega \to [0, \infty]$ such that $Q = Q_a + Q_s$ with

$$Q_a[A] = \int_A Z \, dP, \quad Q_s[A] = Q[A \cap \{Z = \infty\}] \qquad \text{for any } A \in \mathcal{A}.$$

The goal of the exercise is to prove that a Lebesgue density exists if the σ-algebra $\mathcal{A}$ is separable.

a) Show that if Z is a Lebesgue density of Q w.r.t. P then $1/Z$ is a Lebesgue density of P w.r.t. Q. Here $1/\infty := 0$ and $1/0 := \infty$.

From now on assume that the σ-algebra is separable. Hence let $\mathcal{A} = \sigma(\bigcup \mathcal{F}_n)$ where $(\mathcal{F}_n)$ is a filtration consisting of σ-algebras generated by finitely many sets.

b) Write down Lebesgue densities Z_n of Q w.r.t. P on each $\mathcal{F}_n$. Show that

$$Q[Z_n = \infty \text{ and } Z_{n+1} < \infty] = 0 \qquad \text{for any } n,$$

and conclude that (Z_n) is a non-negative supermartingale under P, and $(1/Z_n)$ is a non-negative supermartingale under Q.

c) Prove that the limit $Z_\infty = \lim Z_n$ exists both P-almost surely and Q-almost surely, and $P[Z_\infty < \infty] = 1$ and $Q[Z_\infty > 0] = 1$.

d) Conclude that Z_∞ is a Lebesgue density of P w.r.t. Q on $\mathcal{A}$, and $1/Z_\infty$ is a Lebesgue density of Q w.r.t. P on $\mathcal{A}$.

4. (Uniform integrability). Let (X_n) be a sequence of random variables on a probability space $(\Omega, \mathcal{A}, P)$ auch that $\sup E[|X_n|] < \infty$. Show that:

a) The family $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable if there is an integrable random variable Y such that $|X_n| \leq Y$ for any $n \in \mathbb{N}$.

b) Uniform integrability holds if and only if the measures $Q_n(A) = E[|X_n|\,;\, A]$ are uniformly absolutely continuous w.r.t P, i.e., iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $A \in \mathcal{A}$,

$$P[A] < \delta \quad \Longrightarrow \quad \sup_{n \in \mathbb{N}} E[|X_n|\,;\, A] < \varepsilon.$$

c) If (X_n) converges in L^1 then the family $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable.