Institut für angewandte Mathematik
Winter Semester 15/16
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Introduction to Stochastic Analysis, Problem sheet 5

Please hand in your solutions with your names and the name of your tutor before Tuesday 24.11., 12 am, at the post-boxes opposite to the maths library.

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1. (Uniform integrability).

a) For which sequences (a_n) of real numbers are the random variables

$$X_n \;= a_n \cdot I_{(0,1/n)}\;, \qquad n \in \mathbb{N},$$

uniformly integrable w.r.t the uniform distribution on the interval $(0,1)$?

b) Show that the exponential martingale $M_t = \exp(B_t - t/2)$ of a one-dimensional Brownian motion is not uniformly integrable.

c) Let (M_n) be an $(\mathcal{F}_n)$ martingale with $\sup E[\,|M_n|^p\,] < \infty$ for some $p \in (1,\infty)$. Prove that (M_n) converges almost surely and in L^1, and $M_n = E[M_\infty \mid \mathcal{F}_n]$ for any $n \geq 0$. Hence conclude that $|M_n - M_\infty|^p$ is uniformly integrable, and $M_n \to M_\infty$ in L^p.

2. (Backward Martingale Convergence and Law of Large Numbers). Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a *decreasing* sequence of sub-σ-algebras on a probability space $(\Omega, \mathcal{A}, P)$.

a) Prove that for any random variable $X \in \mathcal{L}^1(\Omega, \mathcal{A}, P)$, the limit $M_{-\infty}$ of the sequence $M_{-n} := E[X \mid \mathcal{F}_n]$ as $n \to \infty$ exists almost surely and in L^1, and

$$M_{-\infty} \;=\; E[X \mid \bigcap \mathcal{F}_n] \qquad \text{almost surely.}$$

b) Now let (X_n) be a sequence of i.i.d. random variables in $\mathcal{L}^1(\Omega, \mathcal{A}, P)$, and let $\mathcal{F}_n = \sigma(S_n, S_{n+1}, \ldots)$ where $S_n = X_1 + \ldots + X_n$. Prove that

$$E[X_1 \mid \mathcal{F}_n] \;=\; \frac{S_n}{n},$$

and conclude that the strong Law of Large Numbers holds:

$$\frac{S_n}{n} \;\longrightarrow\; E[X_1] \qquad \text{almost surely.}$$

3. (Wright model of evolution - Poisson approximation). Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $\mathbb{Z}_+$, initial state $x \in \mathbb{Z}_+$, and transition matrix

$$p(i,k) = e^{-\lambda i}\frac{(\lambda i)^k}{k!} \quad (i \geq 1, k \geq 0), \qquad p(0,0) = 1\,.$$

What is the connection to the Wright model of evolution (Sheet 1, Exercise 4) ?

a) Show that for $\lambda = 1$, the process (X_n) is a martingale w.r.t. P_x, and

$$\lim_{n\to\infty} X_n = 0 \quad P_x\text{-a.s.}.$$

b) Let $T_a := \min\{n \geq 0 : X_n \geq a\}$. Show that for $\lambda = 1$,

$$P_x[T_a < \infty] \;\leq\; x/a.$$

c) What can we conclude in the case $\lambda \neq 1$ using martingale methods?

4. (Local maxima of Brownian paths). Let B_t be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$. Show that the following statements hold for almost every ω:

a) The trajectory $t \mapsto B_t(\omega)$ is not monotone in any interval $[a,b]$ with $a < b$.

b) The set of local maxima of $t \mapsto B_t(\omega)$ is dense in $[0,\infty)$.

c) All local maxima of $t \mapsto B_t(\omega)$ are strict (i.e., for any local maximum m there exists an $\varepsilon > 0$ such that $B_t(\omega) < B_m(\omega)$ for all $t \in (m-\varepsilon, m+\varepsilon)$).