Institut für angewandte Mathematik
Winter Semester 15/16
Andreas Eberle, Raphael Zimmer

Introduction to Stochastic Analysis, Problem sheet 4

Please hand in your solutions with your names and the name of your tutor before Tuesday 17.11., 12 am, at the post-boxes opposite to the maths library.

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1. (Martingales and stopping times of Brownian motion). Let (B_t) be a d-dimensional Brownian motion. Show that:

a) The following processes are martingales w.r.t. each of the filtrations $(\mathcal{F}_t^B)$ and $(\mathcal{F}_t)$, where $\mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^B$ denotes the right continuous filtration :

   i) The coordinate processes $B_t^{(i)}$, $1 \le i \le d$,

   ii) $B_t^{(i)} B_t^{(j)} - t \cdot \delta_{ij}$ for any $1 \le i, j \le d$,

   iii) $\exp(\alpha \cdot B_t - \frac{1}{2}|\alpha|^2 t)$ for any $\alpha \in \mathbb{R}^d$.

b) For a closed set $A \subset \mathbb{R}^d$, the first hitting time $T_A = \inf\{t \ge 0 : B_t \in A\}$ is a stopping time w.r.t. $(\mathcal{F}_t^B)$.

c) For an open set $U \subset \mathbb{R}^d$, T_U is a stopping time w.r.t. $(\mathcal{F}_t)$ but not necessarily w.r.t. $(\mathcal{F}_t^B)$.

2. (Ruin probabilities and passage times revisited). Let (B_t) be a one-dimensional Brownian motion starting at 0. For $a, b > 0$ let

$$T = \inf\{t \ge 0 \;:\; B_t \notin (-b, a)\} \qquad \text{and} \qquad T_a = \inf\{t \ge 0 \;:\; B_t = a\}$$

denote the first exit time from the interval $(-b, a)$, and the first hitting time of a, respectively. You may assume without proof that both stopping times are almost surely finite. Show that:

a) *Ruin probabilities:* $P[B_T = a] = b/(a+b)$, $\quad P[B_T = -b] = a/(a+b)$,

b) *Mean exit time:* $E[T] = a \cdot b$, and $E[T_a] = \infty$,

c) *Laplace transform of passage times:* $E[\exp(-sT_a)] = \exp(-a\sqrt{2s})$ for any $s > 0$.

d) The distribution of T_a on $(0, \infty)$ is absolutely continuous with density

$$f_{T_a}(t) \;=\; a \cdot (2\pi t^3)^{-1/2} \cdot \exp(-a^2/2t).$$

3. (Wiener-Lévy Representation of Brownian Motion).
The Schauder functions $e_{n,k} \in C([0,1])$ are defined in the following way :

$$e_{0,1}(t) := \min(t, 1-t),$$
$$e_{n,k}(t) := \begin{cases} 2^{-n/2} \cdot e_{0,1}(2^n t - k) & \text{for } t \in [k2^{-n}, (k+1)2^{-n}], \\ 0 & \text{otherwise,} \end{cases}$$

$n \in \mathbb{N}$, $k = 0, 1, 2, \ldots, 2^n - 1$. For $x \in C([0,1])$ with $x(0) = 0$ let

$$a_{n,k} \;:=\; 2^{n/2} \cdot \Delta_{n,k} x \qquad \text{with} \qquad \Delta_{n,k} x := 2 \cdot (x(m_{n,k}) - \bar{x}_{n,k}),$$

where $m_{n,k}$ denotes the midpoint of the dyadic interval $[k2^{-n}, (k+1)2^{-n}]$, and

$$\bar{x}_{n,k} \;:=\; (x((k+1) \cdot 2^{-n}) + x(k \cdot 2^{-n}))/2.$$

a) Show that the sequence

$$x_m(t) \;:=\; x(1) \cdot t + \sum_{n=0}^{m} \sum_{k=0}^{2^n-1} a_{n,k} \cdot e_{n,k}(t), \qquad m \in \mathbb{N},$$

converges to $x(t)$ uniformly for $t \in [0,1]$. *(Hint : Verify that x_m is the polygonal approximation of x w.r.t. the m–th dyadic partition of the interval $[0,1]$)*

b) Prove that w.r.t. Wiener measure μ_0 on $\Omega = C([0,1])$, the random variables

$$X_1(\omega) \qquad \text{and} \qquad Y_{n,k}(\omega) \;:=\; 2^{n/2} \cdot \Delta_{n,k} X(\omega) \qquad (n \geq 0, 0 \leq k < 2^n),$$

are independent and standard normally distributed, and the *Wiener–Lévy representation*

$$X_t(\omega) \;=\; X_1(\omega) \cdot t + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} Y_{n,k}(\omega) \cdot e_{n,k}(t) \qquad \text{holds for any } \omega \in \Omega.$$

c) How can this be used in order to simulate sample paths of Brownian motion ?

4. (Bin Packing). Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of i.i.d. random variables taking values in $[0,1]$. How many bins of size 1 are needed to pack n objects of sizes X_1, X_2 , …, X_n ? Let B_n be the minimal number of bins and set

$$M_k \;:=\; E[B_n | \sigma(X_1, \ldots, X_k)]\,, \qquad 0 \leq k \leq n\ .$$

Show that $|M_k - M_{k-1}| \leq 1$ and conclude that

$$P[|B_n - E[B_n]| \geq \varepsilon] \;\leq\; 2 \cdot e^{-\frac{\varepsilon^2}{2n}}\ .$$

Remark: One can show that asymptotically, $E[B_n]$ is growing linearly in n.