Institut für angewandte Mathematik
Winter Semester 15/16
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Introduction to Stochastic Analysis, Problem sheet 3

Please hand in your solutions with your names and the name of your tutor before Tuesday 10.11., 12 am, at the post-boxes opposite to the maths library.

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1. (Integrability of stopping times). *[Murphy's law: Everything that has a realistic chance to happen,will happen — in fact rather earlier than later.]*

a) Prove that the expectation $E[T]$ of a stopping time T is finite if there exist constants $k \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$P\left[T \leq n + k \,|\, \mathcal{F}_n\right] \;\geq\; \varepsilon \qquad P\text{–a.s. for any } n \geq 0.$$

b) Let $S_n = \eta_1 + \eta_2 + \cdots + \eta_n$ be a random walk in $\mathbb{R}^1$ with non-constant i.i.d. increments η_i satisfying $E[\eta_i] = 0$. Prove that the mean exit time of (S_n) from a bounded interval is finite.

2. (Hitting times for the 2-dimensional random walk). Let Z_n be the random walk on $\mathbb{Z}^2$ starting in z_0 and making a step in one of the four directions with equal probability.

a) Show that $|Z_n|^2 - n$ is a martingale.

b) For $r > |z_0|$ let

$$T \;:=\; \inf\{n \geq 0 : |Z_n|^2 \geq r^2\}$$

be the exit time from the circle around 0 with radius r. Prove that

$$r^2 - |z_0|^2 \;\leq\; E[T] \;\leq\; (r+1)^2 - |z_0|^2\,.$$

3. (Random signs). Let (a_n) be a sequence of real numbers with $\sum a_n^2 = \infty$, and let

$$M_n \;=\; \sum_{k=1}^{n} \varepsilon_k a_k\,, \qquad \varepsilon_k \text{ i.i.d. with } \quad P[\varepsilon_k = \pm 1] = 1/2.$$

a) Determine the conditional variance process $\langle M \rangle_n$.

b) For $c > 0$ let $T_c := \inf\{n \geq 0 : |M_n| \geq c\}$. Show that $P[T_c = \infty] = 0$.

c) Conclude that almost surely, the process (M_n) has unbounded oscillations:

$$P[\limsup M_n = +\infty] \;=\; 1 \;=\; P[\liminf M_n = -\infty].$$

4. (Star Trek I). The control system on the star-ship *Enterprise* has gone wonky. All that one can do is to set a distance to be travelled. The spaceship will then move that distance in a randomly chosen direction, then stop. The object is to get into the Solar System, a ball of radius r. Initially, the *Enterprise* is at a distance $R_0 (> r)$ from the Sun.
Let R_n be the distance from Sun to *Enterprise* after n 'space-hops'.

a) Show that, for any strategy which always sets a distance not greater than that from the Sun to the *Enterprise*, $1/R_n$ is a martingale.

   Hint: Use the mean-value property of harmonic functions (Proof?): If $\Delta f = 0$ on a ball $B(x, \mathcal{R}) \subset \mathbb{R}^3$, then the average of f on the sphere $\partial B(x, \mathcal{R})$ is equal to $f(x)$.

b) Deduce that $P[\textit{Enterprise}$ gets into Solar System$] \leq r/R_0$.

c) For each $\varepsilon > 0$, you can choose a strategy which makes this probability greater than $r/R_0 - \varepsilon$. What kind of strategy will that be?

   Hint: You may use without proof that a bounded martingale converges almost surely to a finite limit.