Institut für angewandte Mathematik
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Introduction to Stochastic Analysis, Problem sheet 2

Please hand in your solutions with your names and the name of your tutor before Tuesday 3.11., 12 am, at the post-boxes opposite to the maths library.

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1. (Discretizations of stochastic differential equations). Consider an ordinary differential equation $dX/dt = b(X)$ where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a given vector field. In order to take into account unpredictable effects on a system, one is frequently interested in studying perturbations of the dynamics of type

$$dX_t \quad = \quad b(X_t)\,dt + \text{ “noise”} \qquad t \geq 0, \tag{1}$$

with a random noise term. The solution $(X_t)_{t\geq 0}$ of such a stochastic differential equation (SDE) is a stochastic process in continuous time defined on a probability space $(\Omega, \mathcal{A}, P)$ where also the random variables describing the noise effects are defined. The vector field b is called the (deterministic) “drift”. We will make sense of general SDE later on, but we can already consider time discretizations.

For simplicity let us assume $d = 1$. Let $b, \sigma : \mathbb{R} \to \mathbb{R}$ be continuous functions, and let $(\eta_i)_{i\in\mathbb{N}}$ be a sequence of independent standard normally distributed random variables describing the noise effects. Given an initial value $x_0 \in \mathbb{R}$ and a fine discretization step size $h > 0$, we now define a stochastic process $(X_n^{(h)})$ in discrete time by $X_0^{(h)} = x_0$, and

$$X_{k+1}^{(h)} - X_k^{(h)} \quad = \quad b(X_k^{(h)}) \cdot h + \sigma(X_k^{(h)})\sqrt{h}\,\eta_{k+1}, \qquad \text{for } k = 0, 1, 2, \ldots \tag{2}$$

One should think of $X_k^{(h)}$ as an approximation for the value of the process (X_t) at time $t = k \cdot h$. The equation (2) can be rewritten as

$$X_n^{(h)} \;=\; x_0 + \sum_{k=0}^{n-1} b(X_k^{(h)}) \cdot h + \sum_{k=0}^{n-1} \sigma(X_k^{(h)}) \cdot \sqrt{h} \cdot \eta_{k+1}. \tag{3}$$

To understand the scaling factors h and $\sqrt{h}$ we note first that if $\sigma \equiv 0$ then (2) respectively (3) is the Euler approximation for the o.d.e. $dX/dt = b(X)$. Furthermore, if $b \equiv 0$ and $\sigma \equiv 1$, then the *diffusive scaling* by a factor $\sqrt{h}$ in the second term ensures that as $h \searrow 0$, the continuous time process $X_{\lfloor t/h \rfloor}^{(h)}, t \in [0, \infty)$, converges in distribution to Brownian motion by the functional central limit theorem (Donsker's invariance principle).

a) Prove that $X^{(h)}$ is a time-homogeneous $(\mathcal{F}_n)$ Markov chain with transition kernel

$$p(x, \bullet\,) \quad = \quad N(x + b(x)h, \sigma(x)^2 h)[\,\bullet\,].$$

b) Show that the Doob decomposition $X^{(h)} = M^{(h)} + A^{(h)}$ and the conditional variance process $\langle M^{(h)} \rangle$ of the martingale part are given by

$$A_n^{(h)} = \sum_{k=0}^{n-1} b(X_k^{(h)}) \cdot h, \quad M_n^{(h)} = x_0 + \sum_{k=0}^{n-1} \sigma(X_k^{(h)})\sqrt{h}\,\eta_{k+1}, \quad \langle M^{(h)} \rangle_n = \sum_{k=0}^{n-1} \sigma(X_k^{(h)})^2 \cdot h.$$

2. (Ruin problem for the asymmetric random walk). Let $p \in (0,1)$ with $p \neq 1/2$. We consider the random walk $S_n = Y_1 + \cdots + Y_n$, Y_i $(i \geq 1)$ i.i.d. with $P[Y_i = +1] = p$ and $P[Y_i = -1] = q := 1 - p$.

a) Show that the following processes are martingales:

$$M_n \ := \ (q/p)^{S_n} \,, \qquad N_n \ := \ S_n - n(p-q)\,.$$

b) For $a, b \in \mathbb{Z}$ with $a < 0 < b$ let $T := \min\{n \geq 0 : S_n \notin (a,b)\}$. Deduce from a) that

$$P[S_T = a] \ = \ \frac{1-(p/q)^b}{1-(p/q)^{b-a}}\,, \quad \text{and} \quad E[T] \ = \ \frac{b}{p-q} - \frac{b-a}{p-q} \cdot \frac{1-(p/q)^b}{1-(p/q)^{b-a}}\,.$$

3. (Martingale formulation of Bellman's Optimality Principle). We consider a game consisting of $N \in \mathbb{N}$ rounds. In each round a player can stake an amount C_n satisfying $0 \leq C_n \leq Z_{n-1}$, where Z_{n-1} denotes the player's capital at time $n-1$ and Z_0 is a given positive constant. Let ε_n indicate the winnings per unit stake in the n-th round of the game. Assume that the ε_n are i.i.d. random variables satisfying

$$P[\varepsilon_n = +1] = p\,, \quad P[\varepsilon_n = -1] = q := 1-p\,, \quad \frac{1}{2} < p < 1.$$

Our aim is to maximize the average interest rate $E[\log(Z_N/Z_0)]$. Let

$$M_n := \log Z_n - n\alpha, \qquad \alpha \ := \ p\log p + q \log q + \log 2 \qquad \text{(entropy)}.$$

Show that $(M_n)_{n\in\mathbb{N}}$ is a supermartingale for **any** predictable strategy $(C_n)_{n\in\mathbb{N}}$ and conclude

$$E[\log(Z_N/Z_0)] \leq N\alpha.$$

Find a predictable strategy such that M_n is a martingale. What is the optimal strategy?

4. (CRR model of stock market). Let $\Omega = \{1+a, 1+b\}^N$ with $-1 < a < r < b < \infty$, $X_i(\omega_1, \ldots, \omega_N) = \omega_i$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, and $S_n \ = \ S_0 \cdot \prod_{i=1}^n X_i$, $n = 0, 1, \ldots, N$.

a) *Completeness:* Prove that for any function $F : \Omega \to \mathbb{R}$ there exists a constant V_0 and a predictable sequence $(\Phi_n)_{1\leq n \leq N}$ such that $F = V_N := V_0 + (\Phi_\bullet S)_N$, or, equivalently,

$$(1+r)^{-N} F \ = \ \widetilde{V}_N \ = \ V_0 + (\Phi_\bullet \widetilde{S})_N \qquad \text{where} \qquad \widetilde{S}_n \ := \ (1+r)^{-n} S_n.$$

Hence in the CRR model, any $\mathcal{F}_N$-measurable function F can be replicated by a predictable trading strategy. Market models with this property are called *complete.*

Hint: Prove inductively that for $n = N, N-1, \ldots, 0$, $\tilde{F} = F/(1+r)^N$ can be represented as

$$\tilde{F} = \widetilde{V}_n + \sum\nolimits_{i=n+1}^N \Phi_i \cdot (\widetilde{S}_i - \widetilde{S}_{i-1})$$

with an $\mathcal{F}_n$-measurable function $\widetilde{V}_n$ and a predictable sequence $(\Phi_i)_{n+1 \leq i \leq N}$.

b) *Option pricing:* Derive a general formula for the no-arbitrage price of an option with payoff function $F : \Omega \to \mathbb{R}$ in the CRR model. Compute the no-arbitrage price for a European call option with maturity N and strike K explicitly.