Institut für angewandte Mathematik
Winter Semester 15/16
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Introduction to Stochastic Analysis, Problem sheet 12

Please hand in your solutions with your names and the name of your tutor before Tuesday 02.02., 12 am, at the post-boxes opposite to the maths library.

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1. (Variation of constants). We consider nonlinear stochastic differential equations

$$dX_t = f(t, X_t)\,dt + c(t)X_t\,dB_t, \qquad X_0 = x,$$

where $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R}^+ \to \mathbb{R}$ are continuous (deterministic) functions.

a) Find an explicit solution Z_t of the equation with $f \equiv 0$.

b) To solve the equation in the general case, use the Ansatz $X_t = C_t \cdot Z_t$. Show that the SDE gets the form

$$\frac{dC_t(\omega)}{dt} = f(t, Z_t(\omega) \cdot C_t(\omega))/Z_t(\omega)\ , \qquad C_0 = x. \tag{1}$$

Note that for each $\omega \in \Omega$, this is a *deterministic* differential equation for the function $t \mapsto C_t(\omega)$. We can therefore solve (1) with ω as a parameter to find $C_t(\omega)$.

c) Apply this method to solve the stochastic differential equation

$$dX_t = \frac{1}{X_t}\,dt + \alpha X_t\,dB_t\ , \qquad X_0 = x > 0\ , \qquad \alpha \in \mathbb{R}.$$

d) Apply the method to study the solution of the stochastic differential equation

$$dX_t = X_t^\gamma\,dt + \alpha X_t\,dB_t\ , \qquad X_0 = x > 0\ ,$$

where α and γ are constants. For which values of γ do we get explosion?

2. (Stochastic oscillator).

a) Let A and σ be $d \times d$–matrices, and suppose that (B_t) is a Brownian motion in $\mathbb{R}^d$. Solve the SDE

$$dZ_t = AZ_t\,dt + \sigma\,dB_t\ , \qquad Z_0 = z_0.$$

(First consider $\sigma = 0$, then apply variation of constants)

b) Small displacements from equilibrium (e.g. of a pendulum) with stochastic reset force are described by an SDE of type

$$\begin{aligned} dX_t &= V_t\,dt \\ dV_t &= -X_t\,dt + dB_t \end{aligned}$$

with a one-dimensional Brownian motion (B_t). In complex notation:

$$dZ_t = -iZ_t\,dt + i\,dB_t, \qquad \text{where } Z_t = X_t + iV_t.$$

(i) Solve the SDE with initial conditions $X_0 = x_0$, $V_0 = v_0$.

(ii) Show that X_t is a normally distributed random variable with mean given by the solution of the corresponding deterministic equation.

(iii) Compute the asymptotic variance $\lim_{t\to\infty} \frac{1}{t} \operatorname{Var}[X_t]$.

3. (Cox-Ingersoll-Ross model). Let (B_t) be a Brownian motion. The Cox-Ingersoll-Ross model aims to describe for example an interest rate process (R_t) or a stochastic volatility process and is given by

$$dR_t \;=\; (\alpha - \beta R_t)dt + \sigma\sqrt{R_t}dB_t, \qquad R_0 = x_0 > 0,$$

where $\alpha, \beta, \sigma > 0$. It can be shown that the SDE admits a strong solution.

a) Compute the corresponding *scale function* and study the asymptotic behaviour of R_t depending on the parameters α, β and σ.

b) Suppose that $2\alpha \geq \sigma^2$. We study further properties of R_t :

(i) By applying Itô's formula, show that $E[|R_t|^p] < \infty$ for any $t > 0$ and $p \geq 1$.

(ii) Compute the expectation of R_t. *Hint: Apply Itô's formula to* $f(t,x) = e^{\beta t}x$.

(iii) Proceed in a similar way to compute $\operatorname{Var}[R_t]$, and determine $\lim_{t\to\infty} \operatorname{Var}[R_t]$.

4. (Diffusions in $\mathbb{R}^n$). We consider stochastic differential equations in $\mathbb{R}^n$ of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \qquad X_0 = x \in \mathbb{R}^n,$$

where $b : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are continuous and B_t is a d-dim. Brownian motion for some $d \in \mathbb{N}$. We assume the existence of a strong solution $X_t = (X_t^1, ..., X_t^n)$.

a) Show that the processes

$$M_t^i = X_t^i - X_0^i - \int_0^t b^i(s, X_s)ds, \qquad 1 \leq i \leq n,$$

are local martingales with covariations

$$[M^i, M^j]_s = a_{i,j}(s, X_s) \qquad \text{for any } s \geq 0,\ P\text{-almost surely},$$

where $a(t,x) = (\sigma\sigma^T)(t,x) \in \mathbb{R}^{n\times n}$.

b) Assume that b and σ are bounded. Determine

$$\lim_{t\downarrow 0} \frac{1}{t} E\left[X_t^i - x^i\right] \qquad \text{and} \qquad \lim_{t\downarrow 0} \frac{1}{t} E\left[(X_t^i - x^i)(X_t^j - x^j)\right].$$

c) Show that X_t solves the *time-dependent martingale problem* w.r.t $(\partial_t + \mathscr{L})$, where

$$(\mathscr{L}F)(t,x) = \frac{1}{2}\sum_{i,j=1}^n a_{i,j}(t,x)\frac{\partial^2 F}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^n b^i(t,x)\frac{\partial F}{\partial x_i}(t,x),$$

i.e. show that for any $F \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ the following process is a local martingale :

$$M_t = F(t, X_t) - F(0, X_0) - \int_0^t \left(\frac{\partial F}{\partial t} + \mathscr{L}F\right)(s, X_s)\, ds.$$