**Institut für angewandte Mathematik**
**Winter Semester 15/16**
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# Introduction to Stochastic Analysis, Problem sheet 11

**Please hand in your solutions with your names and the name of your tutor before Tuesday 26.01., 12 am, at the post-boxes opposite to the maths library.**

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**1. (Feynman and Kac at the stock exchange).** The price of a security is modeled by geometric Brownian motion $(X_t)$ with parameters $\alpha, \sigma > 0$. At a price $x$ we have a cost $V(x)$ per unit of time. The total cost up to time $t$ is then given by

$$A_t \;=\; \int_0^t V(X_s)ds\ .$$

Suppose that $u$ is a bounded solution to the PDE

$$\frac{\partial u}{\partial t} \;=\; \mathcal{L}u \;-\; \beta V u\,, \qquad \text{where} \qquad \mathcal{L} \;=\; \frac{\sigma^2}{2}\,x^2\,\frac{d^2}{dx^2} \;+\; \alpha\, x\,\frac{d}{dx}\ .$$

Show that the Laplace transform of $A_t$ is given by $E_x\left[e^{-\beta A_t}\right] = u(t,x)$ .

**2. (Black-Scholes model).** A stock price is modeled by a geometric Brownian Motion $(S_t)$ with parameters $\alpha, \sigma > 0$. We assume that the interest rate is equal to a real constant $r$ for all times. Let $c(t,x)$ be the value of an option at time $t$ if the stock price at that time is $S_t = x$. Suppose that $c(t, S_t)$ is replicated by a hedging portfolio, i.e., there is a trading strategy holding $\phi_t$ shares of stock at time $t$ and putting the remaining portfolio value $V_t - \phi_t S_t$ in the money market account with fixed interest rate $r$ so that the total portfolio value $V_t$ at each time $t$ agrees with $c(t, S_t)$.
"Derive" the *Black-Scholes partial differential equation*

$$\frac{\partial c}{\partial t}(t,x) + rx\frac{\partial c}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c}{\partial x^2}(t,x) \;=\; rc(t,x) \tag{1}$$

and the *delta-hedging rule*

$$\phi_t \;=\; \frac{\partial c}{\partial x}(t, S_t) \qquad (=: \ \text{Delta}\,). \tag{2}$$

*Hint: Consider the discounted portfolio value $\tilde{V}_t = e^{-rt}V_t$ and, correspondingly, $e^{-rt}c(t,S_t)$. Compute the Ito differentials, and conclude that both processes coincide if $c$ is a solution to (1) and $\phi_t$ is given by (2).*

**3. (Lévy's Arcsine law).** State Lévy's Arcsine law for the time $A_t = \int_0^t I_{(0,\infty)}(B_s)ds$ spent by a standard Brownian motion $(B_s)$ in the interval $(0,\infty)$. Prove it by proceeding in the following way :

a) Let $\alpha, \beta > 0$. Show that if $v$ is a bounded solution to the equation

$$\alpha v - \frac{1}{2}v'' + \beta I_{(0,\infty)} v \;=\; 1$$

on $\mathbb{R} \setminus \{0\}$ with $v \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ then

$$v(x) \;=\; E_x\left[\int_0^\infty \exp(-\alpha t - \beta A_t)dt\right] \qquad \text{for any } x \in \mathbb{R}.$$

b) Compute a corresponding solution $v$ and conclude that

$$\int_0^\infty e^{-\alpha t} E_0\left[e^{-\beta A_t}\right] dt \;=\; \frac{1}{\sqrt{\alpha(\alpha+\beta)}} \,.$$

c) Now use the uniqueness of the Laplace inversion to show that the distribution $\mu_t$ of $A_t/t$ under $P_0$ is absolutely continuous with density

$$f_{A_t/t}(s) \;=\; \frac{1}{\pi\sqrt{s(1-s)}} \,.$$

*Hint: You may use without proof that* $\int_0^\infty t^{-1/2}e^{-t}dt = \sqrt{\pi}$.