Institut für angewandte Mathematik
Winter Semester 15/16
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Introduction to Stochastic Analysis, Problem sheet 10

Please hand in your solutions with your names and the name of your tutor before Tuesday 19.01., 12 am, at the post-boxes opposite to the maths library.

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1. (Quadratic variation of Itô integrals). Suppose that $X : [0, \infty) \to \mathbb{R}$ is a continuous function with continuous quadratic variation $[X]$ w.r.t. a fixed sequence (π_n) of partitions s.t. $\mathrm{mesh}(\pi_n) \to 0$.

a) Let $F \in C^1(\mathbb{R})$. Show that the quadratic variation of $t \mapsto F(X_t)$ along (π_n) is given by

$$[F(X)]_t \;=\; \int_0^t F'(X_s)^2 \, d[X]_s.$$

b) Conclude that for $f \in C^1(\mathbb{R})$, the Itô integral $I_t = \int_0^t f(X_s)\, dX_s$ has quadratic variation

$$[I(f)]_t \;=\; \int_0^t f(X_s)^2 \, d[X]_s \, .$$

2. (Complex-valued Brownian motion). A complex-valued Brownian motion is given by $B_t \;=\; B_t^1 \,+\, i\, B_t^2$ with independent one-dimensional Brownian motions (B_t^1) and (B_t^2).

a) Prove that for any holomorphic function F,

$$F(B_t) \;=\; F(B_0) \,+\, \int_0^t F'(B_s)\, dB_s \, ,$$

where F' denotes the complex derivative of F. *Hint: Use the Cauchy-Riemann equations.*

b) Solve the complex-valued SDE $\quad dZ_t \;=\; \alpha\, Z_t\, dB_t \, , \quad \alpha \in \mathbb{C}\, .$

3. (Heat equation on an interval). Let $V : (a, b) \to [0, \infty)$ be continuous and bounded, and suppose that $u \in C^{1,2}((0, \infty) \times (a, b))$ $(-\infty < a < b < \infty)$ is an up to the boundary continuous and bounded solution of the heat equation

$$\frac{\partial u}{\partial t}(t, x) \;=\; \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t, x) \,-\, V(x)\, u(t, x)$$

with initial and boundary conditions

$$u(0, x) = f(x), \quad u(t, a) = h(t), \quad u(t, b) = k(t).$$

By considering an appropriate martingale show that

$$\begin{aligned} u(t,x) &= E_x\left[f(B_t)\,\exp\left(-\int_0^t V(B_s)\,ds\right)\;;\;t\le T_a\wedge T_b\right] \\ &+ E_x\left[h(t-T_a)\,\exp\left(-\int_0^{T_a} V(B_s)\,ds\right)\;;\;T_a<t\wedge T_b\right] \\ &+ E_x\left[k(t-T_b)\,\exp\left(-\int_0^{T_b} V(B_s)\,ds\right)\;;\;T_b<t\wedge T_a\right]. \end{aligned}$$

4. (Lévy Area). If $c(t)=(x(t),y(t))$ is a smooth curve in $\mathbb{R}^2$ with $c(0)=0$, then

$$A(t) = \int_0^t (x(s)y'(s)-y(s)x'(s))\,ds = \int_0^t x\,dy - \int_0^t y\,dx$$

describes the area that is covered by the secant from the origin to $c(s)$ in the interval $[0,t]$. Analogously, for a two-dimensional Brownian motion $B_t=(X_t,Y_t)$ with $B_0=0$, one defines the *Lévy Area*

$$A_t := \int_0^t X_s\,dY_s - \int_0^t Y_s\,dX_s\,.$$

a) Let $\alpha(t)$, $\beta(t)$ be C^1-functions, $p\in\mathbb{R}$, and

$$V_t = ipA_t - \frac{\alpha(t)}{2}\left(X_t^2+Y_t^2\right) + \beta(t)\,.$$

Show that e^{V_t} is a local martingale provided $\alpha'(t)=\alpha(t)^2-p^2$ and $\beta'(t)=\alpha(t)$.

b) Let $t_0\in[0,\infty)$. Show that the solutions of the ordinary differential equations for α and β with $\alpha(t_0)=\beta(t_0)=0$ are

$$\begin{aligned} \alpha(t) &= p\cdot\tanh(p\cdot(t_0-t))\,, \\ \beta(t) &= -\log\cosh(p\cdot(t_0-t))\,. \end{aligned}$$

Hence conclude that

$$E\left[e^{ipA_{t_0}}\right] = \frac{1}{\cosh(pt_0)} \qquad \forall\, p\in\mathbb{R}\,.$$

*c) Show that the distribution of A_t is absolutely continuous with density

$$f_{A_t}(x) = \frac{1}{2t\cosh(\frac{\pi x}{2t})}\,.$$