Institut für angewandte Mathematik
Winter Semester 15/16
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Introduction to Stochastic Analysis, Problem sheet 1

Please hand in your solutions with your names and the name of your tutor before Tuesday 27.10., 2 pm, at the post-boxes opposite to the maths library.

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1. (Martingales). A process $(X_n)_{n\in\mathbb{Z}_+}$ is called *predictable* w.r.t. a filtration $(\mathcal{F}_n)$ iff X_n is measurable w.r.t. $\mathcal{F}_{n-1}$ for any $n \in \mathbb{N}$. Show that:

a) A predictable martingale is almost surely constant.

b) For a nonnegative martingale $(X_n)_{n\geq 0}$ we have almost surely:

$$X_n(\omega) = 0 \quad \Rightarrow \quad X_{n+k}(\omega) = 0 \text{ for any } k \geq 0 \; .$$

2. (Optional stopping). Let $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a filtration. A random variable $T : \Omega \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ is called a *stopping time* iff $\{T \leq n\} \in \mathcal{F}_n$ for any $n \geq 0$.

a) Prove by induction: If (X_n) is a martingale and T is a stopping time w.r.t. $(\mathcal{F}_n)$ then

$$E[X_{T\wedge n}] \;=\; E[X_0] \qquad \forall\, n \geq 0 \; .$$

b) Give sufficient conditions such that $E[X_T] \;=\; E[X_0]$.

3. (Martingales of Random Walks). Let $X_n = x + Y_1 + \ldots + Y_n$ be a simple random walk starting in $x \in \mathbb{Z}$, i.e., the increments Y_i are independent with $P[Y_i = \pm 1] = 1/2$.

a) Let $a(\lambda) := \log\cosh\lambda$. Show that the following processes are martingales :

$$\text{(i) } X_n \qquad \text{(ii) } M_n \;:=\; X_n^2 - n \qquad \text{(iii) } M_n^\lambda \;:=\; e^{\lambda X_n - a(\lambda)n} \text{ for each } \lambda \in \mathbb{R}.$$

b) For $a, b \in \mathbb{Z}$ with $a < x < b$ let $T(\omega) := \min\{n \geq 0 : X_n(\omega) \notin (a, b)\}$. Show by optional stopping that $P[X_T = a] = \frac{b-x}{b-a}$, and derive a formula for $E[T]$.

4. (Wright model of evolution). Consider a population consisting of N individuals with a finite number of possible types in each generation. Each individual selects its type randomly and independently according to the relative frequencies of the types in the previous generation. The model is relevant both for stochastic algorithms and as a basic model in evolutionary biology.

a) Show that the number X_n of individuals of a given type in the n th generation is a martingale.

b) The supermartingale convergence theorem states that a lower bounded supermartingale converges almost surely. Conclude that

$$P\left[X_n = 0 \text{ or } X_n = N \text{ eventually}\right] = 1.$$

The next exercises are optional. You are strongly recommended to do them if you are not very familiar with conditional expectations of random variables given σ-algebras. As background you may consult "Probability with Martingales" by D. Williams, or any standard textbook on measure theoretic probability.

5. (*Revision of conditional expectations 1). *Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\mathcal{F} \subset \mathcal{A}$ a σ-algebra and $X, Y : \Omega \to \mathbb{R}_+$ non-negative random variables. Define the conditional expectation $E[X|\mathcal{F}]$ and show that P-almost surely, the following identities hold:*

a) For $\lambda \in \mathbb{R}$ we have $E[\lambda X + Y|\mathcal{F}] = \lambda E[X|\mathcal{F}] + E[Y|\mathcal{F}]$.

b) $E[E[X|\mathcal{F}]] = E[X]$.

c) If $\sigma(X)$ is independent of $\mathcal{F}$, then $E[X|\mathcal{F}] = E[X]$.

d) Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be measurable spaces. If $Y : \Omega \to S$ is $\mathcal{F}$-measurable, $X : \Omega \to T$ is independent of $\mathcal{F}$ and $f : S \times T \to [0, \infty)$ is product-measurable, then

$$E[f(Y, X)|\mathcal{F}](\omega) = E[f(Y(\omega), X)] \qquad \textit{for } P\textit{-almost every } \omega \in \Omega.$$

6. (*Revision of conditional expectations 2). *Let X, Y, Z be random variables on a joint probability space $(\Omega, \mathcal{A}, P)$. We define*

$$E[X|Y] := E[X|\sigma(Y)].$$

Show the following statements:

a) If $X, Y \in \mathcal{L}^1$ are independent and identically distributed, then P-almost surely,

$$E[X|X+Y] = \frac{1}{2}(X+Y).$$

b) If Z is independent of the pair (X, Y), then P-almost surely,

$$E[X|Y, Z] = E[X|Y].$$

Is this statement still true if we only assume that X and Z are independent?