Institute for Applied Mathematics
Summer semester 2025
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"Introduction to Stochastic Analysis", Sheet 9.

Please hand in your solutions on eCampus by Wednesday, June 18, 10 am.

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1. (Variation of constants). Solve the stochastic differential equation

$$dY_t = r\,dt + \alpha\,Y_t\,dB_t$$

for a one-dimensional Brownian motion (B_t) and constants $r, \alpha \in \mathbb{R}$.
Hint: Consider first the case $r = 0$, then try a variation of constants ansatz.

2. (Itō diffusions). Let $(X_t, \mathbb{P}_x)$ be a solution of the SDE

$$dX_t \;=\; b(X_t)\,dt \;+\; dB_t\;, \qquad X_0 = x \;\; \mathbb{P}_x\text{–a.s.},$$

where (B_t) is a one-dimensional Brownian motion and $b \in C_b(\mathbb{R})$. Prove that:

a) Under $\mathbb{P}_x$, (X_t) solves the *time-dependent martingale problem* for the generator $\mathcal{L} \;=\; \frac{1}{2}\frac{d^2}{dx^2} \;+\; b(x)\frac{d}{dx}$, i.e., for every $u \in C^2([0,\infty) \times \mathbb{R})$,

$$M_t \;:=\; u(t, X_t) - u(0, X_0) - \int_0^t \left(\frac{\partial u}{\partial s} + \mathcal{L}u \right)(s, X_s)\,ds$$

is a local martingale.

b) *Kolmogorov's forward equation* holds, i.e.,

$$\mathbb{E}_x[f(X_t)] \;=\; f(x) + \int_0^t \mathbb{E}_x[\mathcal{L}f(X_s)]\,ds \qquad \forall\, f \in C_b^2(\mathbb{R}).$$

c) $\mu_t[A] := \mathbb{P}_x[X_t \in A]$ is a solution of $\frac{\partial \mu_t}{\partial t} \;=\; \mathcal{L}^*\mu_t$ in the distributional sense, i.e.,

$$\frac{\partial}{\partial t}\int f\,d\mu_t \;=\; \int \mathcal{L}f\,d\mu_t \qquad \forall\, f \in C_b^2(\mathbb{R})\,.$$

d) The function $u(t,x) = \mathbb{E}_x[f(X_t)]$, $f \in C_b(\mathbb{R})$, is the unique bounded solution of

$$\frac{\partial u}{\partial t} \;=\; \mathcal{L}u\;, \qquad u(0,x) = f(x)\,.$$

(You may assume without proof the existence of a solution $u \in C_b^2$, so you only have to prove that such a solution has the stochastic representation stated above)

3. (Recurrence and transience of Brownian motion).

a) Show that for a Brownian motion (B_t) in $\mathbb{R}^2$, every non-empty open ball D is recurrent, i.e.,

$$\mathbb{P}[\forall\, t \geq 0\ \exists\, s \geq t : B_s \in D] \;=\; 1.$$

b) Conclude that a typical Brownian trajectory is dense in $\mathbb{R}^2$.

c) Conversely, show that for $d \geq 3$, every ball in $\mathbb{R}^d$ is transient for Brownian motion, i.e., $|B_t| \to \infty$ almost surely as $t \to \infty$.

4. (Local time of Brownian motion). What happens if we try to apply Itō's formula to $g(B_t)$ when $(B_t)_{t\geq 0}$ is a one dimensional Brownian motion and $g(x) = |x|$?
Since g is not differentiable at 0, we consider the smooth approximations

$$g_\epsilon(x) \;:=\; \begin{cases} |x| & \text{if } |x| \geq \epsilon, \\ \frac{1}{2}(\epsilon + \frac{x^2}{\epsilon}) & \text{if } |x| < \epsilon, \end{cases} \qquad \text{where } \epsilon > 0.$$

a) Show that almost surely,

$$g_\epsilon(B_t) \;=\; g_\epsilon(B_0) \,+\, \int_0^t g_\epsilon'(B_s)\, dB_s \,+\, \frac{1}{2\epsilon}\,\lambda\left[\{s \in [0,t] : B_s \in (-\epsilon, \epsilon)\}\right].$$

b) Prove that as $\epsilon \to 0$,

$$\int_0^t g_\epsilon'(B_s)\, 1_{(-\epsilon,\epsilon)}(B_s)\, dB_s \;\to\; 0$$

in an appropriate sense to be specified.

c) Conclude that almost surely,

$$|B_t| \;=\; |B_0| \,+\, \int_0^t \operatorname{sign}(B_s)\, dB_s \,+\, L_t, \tag{1}$$

where we set $\operatorname{sign}(x) := -1$ for $x \leq 0$ (!) and $\operatorname{sign}(x) := +1$ for $x > 0$, and

$$L_t \;=\; \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\,\lambda\left[\{s \in [0,t] : B_s \in (-\epsilon, \epsilon)\}\right].$$

The process L_t is called the *local time* for Brownian motion at 0.