Institute for Applied Mathematics
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“Introduction to Stochastic Analysis”, Sheet 8.

Please hand in your solutions on eCampus by Wednesday, June 4, 10 am.

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1. (Quadratic variation of stochastic integrals). Let $I_t = \int_0^t H_s \, \mathrm{d}B_s$ where (B_t) is a one-dimensional Brownian motion and $(H_t) \in \mathcal{L}_a^2(0, u; B)$ for every $u \in (0, \infty)$.

a) Show that almost surely,

$$[I]_t \;=\; \int_0^t H_s^2 \, \mathrm{d}s \qquad \text{for all } t \geq 0.$$

b) Conclude that if $|H_t| = 1$ for all $t \geq 0$ then I is a Brownian motion.

c) In general, give a representation of I as a time-changed Brownian motion.

2. (Solutions of SDE). Let (B_t) be a one-dimensional Brownian motion with $B_0 = 0$. Show that the following processes solve the corresponding stochastic differential equations:

a) $X_t = B_t/(1+t)$ solves

$$\mathrm{d}X_t = -(1+t)^{-1} X_t \, \mathrm{d}t + (1+t)^{-1} \, \mathrm{d}B_t \, , \qquad X_0 = 0 \, .$$

b) For $t < \inf\{s > 0 : B_s \notin [-\pi/2, \pi/2]\}$, $X_t = \sin(B_t)$ solves

$$\mathrm{d}X_t = -\frac{1}{2} X_t \, \mathrm{d}t + \sqrt{1 - X_t^2} \, \mathrm{d}B_t \, , \qquad X_0 = 0 \, ,$$

c) $(X_t, Y_t) = (t, e^t B_t)$ solves

$$\begin{bmatrix} \mathrm{d}X_t \\ dY_t \end{bmatrix} = \begin{bmatrix} 1 \\ Y_t \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ e^{X_t} \end{bmatrix} \mathrm{d}B_t \, , \qquad \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

d) $(X_t, Y_t) = (\cosh(B_t), \sinh(B_t))$ solves

$$\begin{bmatrix} \mathrm{d}X_t \\ dY_t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X_t \\ Y_t \end{bmatrix} \mathrm{d}t + \begin{bmatrix} Y_t \\ X_t \end{bmatrix} \mathrm{d}B_t \, , \qquad \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

e) For $x \geq 0$, $X_t = (x^{1/3} + \frac{1}{3} B_t)^3$ solves

$$\mathrm{d}X_t = \frac{1}{3} X_t^{1/3} \, \mathrm{d}t + X_t^{2/3} \, \mathrm{d}B_t \, , \qquad X_0 = x \, .$$

Is the solution of this SDE unique?

3. (Martingales of Brownian motion). Let (B_t) be a one-dimensional Brownian motion with $B_0 = 0$. Show that the following processes are martingales:

a) $X_t = e^{\frac{1}{2}t}\cos(B_t)$, b) $X_t = e^{\frac{1}{2}t}\sin(B_t)$, c) $X_t = (B_t + t)\exp(-B_t - \frac{1}{2}t)$.

4. (Random rotations: Itō vs. Stratonovich). We consider stochastic differential equations of the form

$$\mathrm{d}Z_t \;=\; A\,Z_t\,\mathrm{d}B_t, \qquad Z_0 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \text{where } A = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{1}$$

is the antisymmetric matrix generating the unit rotation in $\mathbb{R}^2$, (B_t) is a *one-dimensional* Brownian motion, and the solution (Z_t) is a stochastic process taking values in $\mathbb{R}^2$.

a) Write down a time-discretisation of the Itō equation (1), and simulate sample paths of the solution.

b) What do you observe? Can you explain your observations?

c) Now consider the Stratonovich equation

$$\circ\,\mathrm{d}Z_t \;=\; AZ_t \circ \mathrm{d}B_t, \qquad Z_0 = \begin{pmatrix}1\\0\end{pmatrix}. \tag{2}$$

Find a numerical discretisation for the SDE and simulate approximate solutions. What do you observe now?

Hint: Make sure that before starting the implementation, you have transformed the discretisation into an accessible form. Matrix inversion in Python:
from scipy import linalg; inversematrix=linalg.inv(matrix)

5. (* Quadratic variation of Brownian motion revisited). Let $(B_t)_{t\geq 0}$ be a Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The goal of this exercise is to show that for an *arbitrary* sequence of partitions such that $\pi_n \subset \pi_{n+1}$ and $\mathrm{mesh}(\pi_n) \to 0$, the quadratic variation $[B]_t$ exists almost surely. W.l.o.g., we assume $B_0 = 0$.

a) Show that for any $0 \leq s \leq s'$, the process $(\tilde{B}_t)$ defined by

$$\tilde{B}_t \;:=\; \begin{cases} B_t & \text{for } t \in [0, s],\\ B_s - (B_t - B_s) & \text{for } t \in [s, s'],\\ B_s - (B_{s'} - B_s) + B_t - B_{s'} & \text{for } t \in [s', \infty), \end{cases}$$

is a Brownian motion, i.e., $(\tilde{B}_t) \sim (B_t)$. *Hint: Exercise 1.*

b) Now fix $t \geq 0$, and let $\mathcal{F}_n$ denote the σ-algebra generated by the random variables $(B_{s'\wedge t} - B_{s\wedge t})^2$, where s and s' are successive partition points in π_m for some $m \geq n$. Show that

$$\sum_{s\in\pi_n}(B_{s'\wedge t} - B_{s\wedge t})^2 \;=\; \mathbb{E}\Big[\sum_{s\in\pi_n}(B_{s'\wedge t} - B_{s\wedge t})^2 \mid \mathcal{F}_n\Big] \;=\; \mathbb{E}\left[B_t^2 \mid \mathcal{F}_n\right].$$

c) Conclude that $\sum_{s\in\pi_n}(B_{s'\wedge t} - B_{s\wedge t})^2$ converges almost surely as $n \to \infty$, and identify the limit.