Institute for Applied Mathematics
Summer semester 2025
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"Introduction to Stochastic Analysis", Sheet 7.

Please hand in your solutions on eCampus by Wednesday, May 28, 10 am.

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1. (Time-dependent Itō formula). Suppose that $X\colon [0,\infty) \to \mathbb{R}$ is a continuous function with continuous quadratic variation $[X]$ w.r.t. a sequence $(\pi_n)_{n\in\mathbb{N}}$ of partitions s.t. $\mathrm{mesh}(\pi_n) \to 0$. Show that for every function $F \in C^2(\mathbb{R}^2)$ and for every $t \in [0,\infty)$, the Itō integral

$$\int_0^t \frac{\partial F}{\partial x}(s, X_s)\,\mathrm{d}X_s \;=\; \lim_{n\to\infty} \sum_{s\in\pi_n} \frac{\partial F}{\partial x}(s, X_s)\,(X_{s'\wedge t} - X_{s\wedge t})$$

exists, and the time-dependent Itō formula

$$F(t, X_t) - F(0, X_0) = \int_0^t \frac{\partial F}{\partial s}(s, X_s)\,\mathrm{d}s + \int_0^t \frac{\partial F}{\partial x}(s, X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s)\,\mathrm{d}[X]_s \quad (1)$$

holds.

Hint: You may assume without proof that by Taylor's formula, there exists a function $o\colon [0,\infty) \to [0,\infty)$ satisfying $o(r)/r \to 0$ as $r \to 0$, such that for any $s, s' \in [0,t]$,

$$\begin{aligned} F(s', X_{s'}) - F(s, X_s) \;&=\; \frac{\partial F}{\partial s}(s, X_s)\,\delta s \;+\; \frac{\partial F}{\partial x}(s, X_s)\,\delta X_s \;+\; \frac{1}{2}\frac{\partial^2 F}{\partial s^2}(s, X_s)\,(\delta s)^2 \\ &\quad + \frac{\partial^2 F}{\partial s \partial x}(s, X_s)\,\delta s\,\delta X_s \;+\; \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(s, X_s)\,(\delta X_s)^2 \;+\; o\left((\delta s)^2 + (\delta X_s)^2\right). \end{aligned}$$

2. (Stochastic integrals w.r.t. Itō processes). Let

$$I_s \;:=\; \int_0^s H_r\,\mathrm{d}B_r, \qquad 0 \le s \le t,$$

with an $(\mathcal{F}_s)$-Brownian motion B on $(\Omega, \mathcal{A}, \mathbb{P})$, and a process $H \in \mathcal{L}^2_a(0,t;B)$. Suppose that $(\pi_n)_{n\in\mathbb{N}}$ is a sequence of partitions of $[0,t]$ such that $\mathrm{mesh}(\pi_n) \to 0$.

Prove that if G is an $(\mathcal{F}_s)$-adapted bounded continuous process, then the Riemann sums $\sum_{s\in\pi_n} G_s \cdot (I_{s'} - I_s)$ converge in $L^2(\mathbb{P})$, and

$$\int_0^t G_s\,\mathrm{d}I_s \;=\; \lim_{n\to\infty} \sum_{s\in\pi_n} G_s \cdot (I_{s'} - I_s) \;=\; \int_0^t G_s\,H_s\,\mathrm{d}B_s\;.$$

Hint: Express the Riemann sums as a stochastic integral $\int_0^t \ldots \mathrm{d}B_s$ w.r.t. Brownian motion.

3. (Geometric Brownian motion). Let $(B_s)_{s\geq 0}$ be an $(\mathcal{F}_s)$-Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$ s.t. $B_0 = 0$. A geometric Brownian motion $(X_s)_{s\geq 0}$ with parameters $\mu, \alpha \in \mathbb{R}$ is a solution of the stochastic differential equation (SDE)

$$\mathrm{d}X_t \;=\; \mu\, X_t\, \mathrm{d}t \;+\; \alpha\, X_t\, \mathrm{d}B_t\,,$$

i.e., $(X_s)_{s\geq 0}$ is an almost surely continuous and $(\mathcal{F}_s)$ adapted process such that $\mathbb{P}$-almost surely,

$$X_t - X_0 \;=\; \mu \int_0^t X_s\, \mathrm{d}s + \alpha \int_0^t X_s\, \mathrm{d}B_s \qquad \text{for any } t \geq 0.$$

a) Find a solution of the SDE with initial value $X_0 = x_0$ using the ansatz

$$X_t \;=\; x_0 \cdot \exp(aB_t + bt)\,.$$

Here you may assume the time-dependent Itō formula (1).

b) What can you say about the asymptotic behaviour of the process as $t \to \infty$?

c) Compute $\mathbb{E}[X_t]$ and $\mathrm{Cov}[X_s, X_t]$ for $s, t \geq 0$.

4. (Progressive measurability). Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration.

a) Prove that an $(\mathcal{F}_t)$ adapted left-continuous stochastic process $(X_t)_{t\in[0,\infty)}$ is $(\mathcal{F}_t)$ progressively measurable.

b) Show that if $(X_t)_{t\in[0,\infty)}$ is a progressively measurable process and $T\colon \Omega \to [0,\infty]$ is an $(\mathcal{F}_t)$ stopping time then the random variable $X_T \cdot 1_{T<\infty}$ is measurable w.r.t. $\mathcal{F}_T$.