Institute for Applied Mathematics
Summer semester 2025
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"Introduction to Stochastic Analysis", Sheet 6.

Please hand in your solutions on eCampus by Wednesday, May 21, 10 am.

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1. (Local martingales). Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Show that if $(M_t)_{t\in[0,\infty)}$ is an $(\mathcal{F}_t)$ martingale on $(\Omega, \mathcal{A}, \mathbb{P})$ and $T : \Omega \to [0,\infty]$ is predictable, then $(M_t)_{t\in[0,T)}$ is a local martingale up to T.

Now suppose conversely that (M_t) is a continuous local martingale up to $T = \infty$.

b) Show that (M_t) is a martingale if and only if there exists a localizing sequence $(T_k)_{k\in\mathbb{N}}$ such that for every $t \in [0,\infty)$, the family of random variables $\{M_{t\wedge T_k} : k \in \mathbb{N}\}$ is uniformly integrable.

c) Verify that $T_k := \inf\{t \geq 0 : |M_t| \geq k\}$ is a localizing sequence for which the stopped processes $(M_{t\wedge T_k})_{t\geq 0}$ are bounded martingales in $\mathcal{M}_c^2([0,\infty))$.

2. (A local martingale that is not a martingale). Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^3$ with initial value $B_0 = x$, $x \neq 0$. Show that:

a) $X_t = 1/\|B_t\|$ is a local martingale up to $T = \inf\{t \geq 0 : B_t = 0\}$.

b) $T = \infty$ almost surely.

c) $\{X_s : 0 \leq s \leq t\}$ is uniformly integrable for all $t \geq 0$.

d) X_t is *not* a martingale.

Hint: You may assume without proof the multi-dimensional Itô formula for Brownian motion: If U is an open subset of $\mathbb{R}^d$, then for $F \in C^2(U)$,

$$F(B_t) - F(B_0) \;=\; \sum_{i=1}^{d} \int_0^t \frac{\partial F}{\partial x^i}(B_s)\, dB_s^i + \frac{1}{2}\int_0^t \Delta F(B_s)\, ds \qquad \forall\, t < T_{U^C},$$

where $T_{U^C} = \inf\{t \geq 0 : B_t \notin U\}$ is the first exit time from U.

3. (Uniqueness of the angle bracket process). Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Suppose that (M_t) is a square integrable continuous $(\mathcal{F}_t)$ martingale such that for every $t \in \mathbb{R}_+$, the first variation

$$V_t^{(1)}(M) \;=\; \sup_\pi \sum_{s\in\pi} |M_{s'\wedge t} - M_{s\wedge t}|$$

is an almost surely bounded random variable. Show that $t \mapsto M_t$ is almost surely constant.

Hint: $\mathbb{E}[(M_t - M_0)^2] = \sum_{s\in\pi} \mathbb{E}[(M_{s'\wedge t} - M_{s\wedge t})^2]$.

b) More generally, prove that a continuous local martingale M with almost surely finite variation paths is almost surely constant.

c) Conclude that the angle bracket process $\langle M\rangle$ of a continuous local martingale is uniquely determined up to modification on a measure zero set.