Institute for Applied Mathematics
Summer semester 2025
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"Introduction to Stochastic Analysis", Sheet 5.

Please hand in your solutions on eCampus by Wednesday, May 14, 10 am.

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1. (Itō and Stratonovich integrals). Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion starting at 0 on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let $\pi_n = \{k2^{-n} : k \in \mathbb{N}_0\}$ be the n-th dyadic partition of $[0, \infty)$.

a) Show that for any $t \geq 0$,

$$\lim_{n\to\infty} \sum_{s\in\pi_n} (B_{s'\wedge t} - B_{s\wedge t})^2 \;=\; \lim_{n\to\infty} \sum_{s\in\pi_n} (s' \wedge t - s \wedge t) \;=\; t \quad \text{in } L^2(\mathbb{P})\,,$$

and for $p > 2$,

$$\lim_{n\to\infty} \sum_{s\in\pi_n} |B_{s'\wedge t} - B_{s\wedge t}|^p \;=\; 0 \quad \text{in } L^2(\mathbb{P})\,.$$

b) The Stratonovich integral of a process (H_s) w.r.t. (B_s) over $[0, t]$ is defined by

$$\int_0^t H_s \circ \mathrm{d}B_s \;:=\; \lim_{n\to\infty} \sum_{s\in\pi_n} \frac{1}{2}(H_{s'\wedge t} + H_{s\wedge t}) \cdot (B_{s'\wedge t} - B_{s\wedge t})$$

if the limit exists in $L^2(\mathbb{P})$. Show that

$$\int_0^t B_s \circ \mathrm{d}B_s \;=\; \frac{1}{2}B_t^2 \qquad \text{and} \qquad \int_0^t B_s \,\mathrm{d}B_s \;=\; \frac{1}{2}B_t^2 - \frac{1}{2}t\,.$$

c) More generally, prove that for every $m \in \mathbb{N} \setminus \{1\}$,

$$B_t^m \;=\; m\int_0^t B_s^{m-1} \circ \mathrm{d}B_s \;=\; m\int_0^t B_s^{m-1} \,\mathrm{d}B_s + \frac{1}{2}m(m-1)\int_0^t B_s^{m-2} \,\mathrm{d}s.$$

Hint: You can use the identity

$$(x+h)^m - x^m \;=\; mx^{m-1}h + \binom{m}{2}x^{m-2}h^2 + O(|h|^3).$$

2. (Stieltjes integrals).

a) State the definition of the Lebesgue-Stieltjes integral

$$\int_0^t f(s)\,\mathrm{d}g(s)$$

of a locally bounded measurable function $f\colon [0,\infty) \to \mathbb{R}$ w.r.t. a non-decreasing continuous function $g\colon [0,\infty) \to \mathbb{R}$.

b) The *variation* of a function $g\colon [0,\infty) \to \mathbb{R}$ on the interval $[0,t]$ is defined by

$$V^{(1)}(t) \;:=\; \sup_\pi \sum_{s\in\pi} |g(s'\wedge t) - g(s\wedge t)|\,,$$

where the supremum is taken over all partitions of $[0,\infty)$. Show that for continuous functions g with finite variation, both $V^{(1)}$ and $V^{(1)} - g$ are non-decreasing and continuous. Use this fact to extend the definition of the Lebesgue-Stieltjes integral to continuous integrators g of finite variation.

c) Let (π_n) be a sequence of partitions of $[0,\infty)$ with $\mathrm{mesh}(\pi_n) \to 0$, and suppose that $f, g\colon [0,\infty) \to \mathbb{R}$ are continuous functions. Show that if g has finite variation, then the Riemann-Stieltjes integral

$$\int_0^t g(s)\,\mathrm{d}f(s) \;:=\; \lim_{n\to\infty} \sum_{s\in\pi_n} g(s)\,(f(s'\wedge t) - f(s\wedge t))$$

exists, and the integration by parts identity

$$\int_0^t f(s)\,\mathrm{d}g(s) \;=\; f(t)g(t) \;-\; f(0)g(0) \;-\; \int_0^t g(s)\,\mathrm{d}f(s)$$

holds. In particular, $\int g\,\mathrm{d}f$ is independent of the choice of the partition sequence.

3. (Simulation of stochastic integrals). Let (B_t) be a one-dimensional Brownian motion starting at 0 on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Use Riemann sum approximations to simulate the stochastic processes

$$I_t \;=\; \int_0^t B_s\,\mathrm{d}B_s \qquad \text{and} \qquad \hat{I}_t \;=\; \int_0^t B_s\,\hat{\mathrm{d}}B_s \qquad \text{for } t\in[0,1].$$

Here the first integral is an Itō integral, and the second integral is a backward Itō integral.

b) Plot the graphs of samples from the difference process $\hat{I}_t - I_t$. What do you observe? State a conjecture.

c) Can you prove your conjecture?