Institute for Applied Mathematics
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"Introduction to Stochastic Analysis", Sheet 4.

Please hand in your solutions on eCampus by Wednesday, May 7, 10 am.

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1. (Passage probabilities).

a) Let M be a non-negative martingale with continuous sample paths such that $M_0 = x > 0$. Assume that $M_t \to 0$ a.s. as $t \to \infty$. Show that, for every $y > x$,

$$\mathbb{P}\left[\sup_{t\geq 0} M_t \geq y\right] = \frac{x}{y}.$$

b) Show that for a one-dimensional Brownian motion $(B_t)_{t\geq 0}$ with $B_0 = 0$ and $m > 0$,

$$\sup_{t\geq 0}(B_t - mt)$$

is exponentially distributed with parameter $2m$.

c) Compute the passage probabilities of Brownian motion for arbitrary lines $t \mapsto a + mt$ with $a, m \in \mathbb{R}$.

2. (Backward martingale convergence and the law of large numbers). Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a *decreasing* sequence of sub-σ-algebras on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) Prove that for every random variable $X \in \mathcal{L}^1(\Omega, \mathcal{A}, \mathbb{P})$, the limit $M_{-\infty}$ of the sequence $M_{-n} := \mathbb{E}[X \mid \mathcal{F}_n]$ as $n \to \infty$ exists almost surely and in L^1, and

$$M_{-\infty} \;=\; \mathbb{E}[X \mid \bigcap \mathcal{F}_n] \qquad \text{almost surely.}$$

Hint: Apply Doob's upcrossing inequality to the martingales $(M_{k-n})_{k=0,1,\dots n}$.

b) Now let (X_n) be a sequence of i.i.d. random variables in $\mathcal{L}^1(\Omega, \mathcal{A}, \mathbb{P})$, and let $\mathcal{F}_n = \sigma(S_n, S_{n+1}, \dots)$, where $S_n = X_1 + \dots + X_n$. Prove that almost surely,

$$\mathbb{E}[X_1 \mid \mathcal{F}_n] \;=\; \frac{S_n}{n},$$

and conclude that the strong Law of Large Numbers holds:

$$\frac{S_n}{n} \longrightarrow \mathbb{E}[X_1] \qquad \text{almost surely.}$$

3. (Martingale proof of Radon-Nikodym Theorem). Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures on $(\Omega, \mathcal{A})$ such that $\mathbb{Q}$ is *absolutely continuous w.r.t.* $\mathbb{P}$, i.e., every $\mathbb{P}$-measure zero set is also a $\mathbb{Q}$-measure zero set. A *relative density* of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on a sub-σ-algebra $\mathcal{F} \subseteq \mathcal{A}$ is an $\mathcal{F}$-measurable random variable $Z : \Omega \to [0, \infty)$ such that

$$\mathbb{Q}[A] \;=\; \int_A Z \, \mathrm{d}\mathbb{P} \qquad \text{for any } A \in \mathcal{F}.$$

The goal of the exercise is to prove that a relative density on the σ-algebra $\mathcal{A}$ exists if it is separable. Hence let $\mathcal{A} = \sigma(\bigcup \mathcal{F}_n)$ where $(\mathcal{F}_n)$ is a filtration consisting of σ-algebras $\mathcal{F}_n$ that are generated by finitely many disjoints sets $B_{n,i}$ $(i = 1, \ldots, k_n)$ such that $\bigcup_i B_{n,i} = \Omega$.

a) Write down explicitly relative densities Z_n of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on each $\mathcal{F}_n$, and show that (Z_n) is a non-negative martingale under $\mathbb{P}$.

b) Prove that the limit $Z_\infty = \lim Z_n$ exists both $\mathbb{P}$-almost surely and in $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

c) Conclude that Z_∞ is a relative density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on $\mathcal{A}$.

4. (Simulation of Ornstein-Uhlenbeck processes II). A two-dimensional *Ornstein-Uhlenbeck process* is a stochastic process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}^2$ that solves a stochastic differential equation $\mathrm{d}X_t = AX_t \, \mathrm{d}t + \sigma \, \mathrm{d}B_t$, $X_0 = x_0$, i.e.

$$X_t \;=\; x_0 \;+\; \int_0^t A\, X_s \, \mathrm{d}s \;+\; \sigma\, B_t \qquad \text{for all } t \in [0, \infty), \tag{1}$$

where $(B_t)_{t\geq 0}$ is a two-dimensional Brownian motion, A is a 2×2 matrix, and $\sigma \in (0, \infty)$ and the initial value $x_0 \in \mathbb{R}^2$ are given constants.

a) Write down a time-discretization of (1), where $t \in h\mathbb{Z}_+$ for a given step size $h > 0$.

b) Simulate a sample path of a general two dimensional Ornstein Uhlenbeck process on a time interval $[0, t_{\max}]$, and plot the trajectory.

c) Run the simulation for the following choices of A and σ, $t_{\max} = 40$ and $x_0 = (1, 0)$.

   (i) *Two dimensional Brownian motion:* $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma = 1$.

   (ii) *Standard two dimensional OU process:* $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma = 1$.

   (iii) *Randomly perturbed rotation:* $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma = 0.1$ and $\sigma = 1$.

   (iv) *Randomly perturbed rotation with damping:* $A = \begin{pmatrix} -\gamma & 1 \\ -1 & -\gamma \end{pmatrix}$, $\sigma, \gamma \in \{0.1, 1\}$.

   (v) *Damping in one component:* $A = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix}$, $\sigma, \gamma \in \{0.1, 1\}$.

Hint: It might make sense to choose a higher resolution and thin lines for the plots. For example in Python if the numerical solution is stored in a $2 \times$ steps *array* sde:

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plt.figure(figsize=(7,7), dpi=500)
plt.plot(sde[0],sde[1],linewidth=.2)
plt.show()
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