Institute for Applied Mathematics
Summer semester 2025
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"Introduction to Stochastic Analysis", Sheet 12.

Please hand in your solutions on eCampus by Wednesday, July 9, 10 am.

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1. (Stochastic oscillator).

a) Let A and σ be $d \times d$–matrices, let $a \in \mathbb{R}^d$, and suppose that $(B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$.

(i) Solve the SDE

$$dZ_t \;=\; (AZ_t + a)\,dt \;+\; \sigma\,dB_t\;, \qquad Z_0 = z_0.$$

(ii) Show that Z_t is a normally distributed random vector with mean vector $m(t)$ and covariance matrix $C(t)$ where m and C are solutions of the ordinary differential equations

$$\dot{m} = Am + a, \qquad \dot{C} = AC + CA^T + \sigma\sigma^T.$$

b) Small displacements from equilibrium (e.g. of a pendulum) with stochastic reset force and friction coefficient γ are described by an SDE of type

$$\begin{aligned} dX_t &= V_t\,dt \\ dV_t &= -X_t\,dt \;-\gamma V_t\,dt \;+\; dB_t \end{aligned}$$

with a one-dimensional Brownian motion $(B_t)_{t\geq 0}$.

(i) Solve the SDE with initial conditions $X_0 = x_0$, $V_0 = v_0$.

(ii) Show that X_t is a normally distributed random variable with mean given by the solution of the corresponding deterministic equation.

(iii) Show that if $\gamma > 0$ and (X_0, V_0) has an appropriate Gaussian distribution then (X_t, V_t) is a stationary Gaussian process.

2. (Lévy characterizations and random rotations).

a) Suppose that $(M_t^i)_{t\geq 0}$, $i = 1, \ldots, d$, are continuous local martingales with covariations $[M^i, M^j]_t = \delta_{i,j} t$. Show that $M_t = (M_t^1, \ldots, M_t^d)$ is a d-dimensional Brownian motion.
Hint: Try to argue similarly as in the one-dimensional case.

b) Let $(B_t)_{t\geq 0}$ be a d-dimensional Brownian motion, and suppose that $(O_t)_{t\geq 0}$ is a continuous adapted process taking values in the orthogonal $d \times d$ matrices. Prove that the process

$$X_t \;=\; \int_0^t O_s\,dB_s$$

is again a d-dimensional Brownian motion.

3. (Translations of normal distributions).

a) Let $C \in \mathbb{R}^{n\times n}$ be a symmetric non-negative definite matrix, and let $h \in \mathbb{R}^n$.

(i) Show that if C is non-degenerate then $\mathcal{N}(h,C)$ and $\mathcal{N}(0,C)$ are mutually absolutely continuous with relative density

$$\frac{d\mathcal{N}(h,C)}{d\mathcal{N}(0,C)}(x) \;=\; e^{(h,x)-\frac{1}{2}(h,h)} \qquad \text{for } x \in \mathbb{R}^n, \tag{1}$$

where $(g,h) := g \cdot C^{-1}h$ for $g,h \in \mathbb{R}^n$.

(ii) Prove that in general, $\mathcal{N}(h,C)$ is absolutely continuous w.r.t. $\mathcal{N}(0,C)$ if and only if h is orthogonal to the kernel of C w.r.t. the Euclidean inner product.

b) Now consider the probability measures

$$\mathbb{P} \;=\; \bigotimes_{i=1}^{\infty} \mathcal{N}(a_i,1) \quad \text{and} \quad \mathbb{Q} \;=\; \bigotimes_{i=1}^{\infty} \mathcal{N}(0,1)$$

on $\mathbb{R}^{\mathbb{N}}$ endowed with the product σ-algebra, where $(a_i)_{i\in\mathbb{N}}$ is a sequence of real numbers. Let $\mathcal{F}_n = \sigma(X_1,\ldots,X_n)$ where X_k is the evaluation of the k-th coordinate.

(i) Show that $\mathbb{P}$ is absolutely continuous w.r.t. $\mathbb{Q}$ on $\mathcal{F}_n$ with relative density

$$Z_n \;=\; \prod_{i=1}^{n} \exp\left(a_i X_i - a_i^2/2\right).$$

(ii) Now assume that $\sum_{i=1}^{\infty} a_i^2 < \infty$. Show that in this case, $M_n = \sqrt{Z_n}/\mathbb{E}\left[\sqrt{Z_n}\right]$ is an L^2 bounded martingale and Z_n is a uniformly integrable martingale. Conclude that $\mathbb{P}$ is absolutely continuous w.r.t. $\mathbb{Q}$ on the product σ-algebra $\mathcal{F}_\infty$.

(*iii) Conversely, show that $\mathbb{P}$ is not absolutely continuous w.r.t. $\mathbb{Q}$ if $\sum a_i^2 = \infty$.