Institute for Applied Mathematics
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“Introduction to Stochastic Analysis”, Sheet 11.

Please hand in your solutions on eCampus by Wednesday, July 2, 10 am.

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1. (Cox-Ingersoll-Ross model). Let $(B_t)_{t\geq 0}$ be a Brownian motion. The Cox-Ingersoll-Ross model aims to describe, for example, an interest rate process $(R_t)_{t\geq 0}$ or a stochastic volatility process and is given by

$$dR_t \;=\; (\alpha - \beta R_t)dt + \sigma\sqrt{R_t}dB_t, \qquad R_0 = x_0 > 0,$$

where $\alpha, \beta, \sigma > 0$. It can be shown that the SDE admits a strong solution.

a) Compute the corresponding *scale function* and study the asymptotic behaviour of R_t depending on the parameters α, β and σ.

b) Suppose that $2\alpha \geq \sigma^2$. We study further properties of R_t :

   (i) By applying Itō's formula, show that $\mathbb{E}[|R_t|^p] < \infty$ for any $t > 0$ and $p \geq 1$.

   (ii) Compute the expectation of R_t. *(Hint: Apply Itō's formula to $f(t,x) = e^{\beta t}x$.)*

   (iii) Proceed in a similar way to compute $\mathrm{Var}[R_t]$, and determine $\lim_{t\to\infty} \mathrm{Var}[R_t]$.

2. (Black-Scholes model). A stock price is modeled by a geometric Brownian Motion $(S_t)_{t\geq 0}$ with parameters $\alpha, \sigma > 0$. We assume that the interest rate is equal to a real constant r for all times. Let $c(t,x)$ be the value of an option at time t if the stock price at that time is $S_t = x$. Suppose that $c(t,S_t)$ is replicated by a hedging portfolio, i.e., there is a trading strategy holding ϕ_t shares of stock at time t and putting the remaining portfolio value $V_t - \phi_t S_t$ in the money market account with fixed interest rate r so that the total portfolio value V_t at each time t agrees with $c(t,S_t)$.
“Derive” the *Black-Scholes partial differential equation*

$$\frac{\partial c}{\partial t}(t,x) + rx\frac{\partial c}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c}{\partial x^2}(t,x) \;=\; rc(t,x) \tag{1}$$

and the *delta-hedging rule*

$$\phi_t \;=\; \frac{\partial c}{\partial x}(t,S_t) \qquad (=:\ \text{Delta}). \tag{2}$$

(Hint: Consider the discounted portfolio value $\tilde{V}_t = e^{-rt}V_t$ and, correspondingly, $e^{-rt}c(t,S_t)$. Compute the Itō differentials, and conclude that both processes coincide if c is a solution to (1) and ϕ_t is given by (2).)

3. (Variation of constants II). We consider nonlinear stochastic differential equations

$$dX_t = f(t, X_t)\,dt + c(t)X_t\,dB_t, \qquad X_0 = x,$$

where $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R}^+ \to \mathbb{R}$ are continuous (deterministic) functions.

a) Find an explicit solution Z_t of the equation with $f \equiv 0$.

b) To solve the equation in the general case, use the Ansatz $X_t = C_t \cdot Z_t$. Show that the SDE gets the form

$$\frac{dC_t(\omega)}{dt} \;=\; f(t, Z_t(\omega) \cdot C_t(\omega))/Z_t(\omega)\ , \qquad C_0 = x. \tag{3}$$

Note that for each $\omega \in \Omega$, this is a *deterministic* differential equation for the function $t \mapsto C_t(\omega)$. We can therefore solve (3) with ω as a parameter to find $C_t(\omega)$.

c) Apply the method to study the solution of the stochastic differential equation

$$dX_t = X_t^\gamma\,dt + X_t\,dB_t\ , \qquad X_0 = x > 0\ ,$$

where γ is a constant. For which values of γ does the solution explode in finite time?

4. (Lévy Area). If $c(t) = (x(t), y(t))$ is a smooth curve in $\mathbb{R}^2$ with $c(0) = 0$, then

$$A(t) \;=\; \int_0^t (x(s)y'(s) - y(s)x'(s))\,ds \;=\; \int_0^t x\,dy - \int_0^t y\,dx$$

describes the area that is covered by the secant from the origin to $c(s)$ in the interval $[0, t]$. Analogously, for a two-dimensional Brownian motion $B_t = (X_t, Y_t)$ with $B_0 = 0$, one defines the *Lévy Area*

$$A_t \;:=\; \int_0^t X_s\,dY_s \;-\; \int_0^t Y_s\,dX_s\,.$$

a) Let $\alpha(t)$, $\beta(t)$ be C^1-functions, $p \in \mathbb{R}$, and

$$V_t \;=\; ipA_t \;-\; \frac{\alpha(t)}{2}\left(X_t^2 + Y_t^2\right) \;+\; \beta(t)\,.$$

Show that e^{V_t} is a local martingale provided $\alpha'(t) = \alpha(t)^2 - p^2$ and $\beta'(t) = \alpha(t)$.

b) Let $t_0 \in [0, \infty)$. Show that the solutions of the ordinary differential equations for α and β with $\alpha(t_0) = \beta(t_0) = 0$ are

$$\begin{aligned} \alpha(t) &= p \cdot \tanh(p \cdot (t_0 - t))\,, \\ \beta(t) &= -\log\cosh(p \cdot (t_0 - t))\,. \end{aligned}$$

Hence conclude that

$$\mathbb{E}\left[e^{ipA_{t_0}}\right] \;=\; \frac{1}{\cosh(pt_0)} \qquad \forall\, p \in \mathbb{R}\,.$$

c) Show that the distribution of A_t is absolutely continuous with density

$$f_{A_t}(x) \;=\; \frac{1}{2t\cosh(\frac{\pi x}{2t})}\,.$$