Institute for Applied Mathematics
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"Introduction to Stochastic Analysis", Sheet 10.

Please hand in your solutions on eCampus by Wednesday, June 25, 10 am.

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1. (Complex-valued Brownian motion). A complex-valued Brownian motion is given by $B_t = B_t^1 + iB_t^2$ with independent one-dimensional Brownian motions B^1 and B^2.

a) Prove that for any holomorphic function F,

$$F(B_t) \;=\; F(B_0) + \int_0^t F'(B_s)\, dB_s \;,$$

where F' denotes the complex derivative of F.
Hint: Use the Cauchy-Riemann equations.

b) Solve the complex-valued SDE $\quad dZ_t \;=\; \alpha\, Z_t\, dB_t, \quad \alpha \in \mathbb{C}\,.$

2. (Heat equation on an interval). Let $V : (a,b) \to [0,\infty)$ be continuous and bounded, and suppose that $u \in C^{1,2}([0,\infty) \times (a,b))$ $(-\infty < a < b < \infty)$ is an up to the boundary continuous and bounded solution of the heat equation

$$\frac{\partial u}{\partial t}(t,x) \;=\; \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) \;-\; V(x)\, u(t,x)$$

with initial and boundary conditions

$$u(0,x) = f(x), \quad u(t,a) = h(t), \quad u(t,b) = k(t).$$

By considering an appropriate martingale show that

$$\begin{aligned} u(t,x) \;=\; & \mathbb{E}_x\left[f(B_t)\, \exp\left(-\int_0^t V(B_s)\, ds \right) \;;\; t \le T_a \wedge T_b \right] \\ & + \mathbb{E}_x\left[h(t - T_a)\, \exp\left(-\int_0^{T_a} V(B_s)\, ds \right) \;;\; T_a < t \wedge T_b \right] \\ & + \mathbb{E}_x\left[k(t - T_b)\, \exp\left(-\int_0^{T_b} V(B_s)\, ds \right) \;;\; T_b < t \wedge T_a \right] . \end{aligned}$$

3. (Feynman and Kac at the stock exchange). The price of a security is modeled by geometric Brownian motion (X_t) with parameters $\alpha, \sigma > 0$. At a price x we have a cost $V(x)$ per unit of time. The total cost up to time t is then given by

$$A_t = \int_0^t V(X_s) ds .$$

Suppose that u is a bounded solution to the PDE

$$\frac{\partial u}{\partial t} = \mathcal{L}u - \beta V u , \qquad \text{where} \qquad \mathcal{L} = \frac{\sigma^2}{2} x^2 \frac{d^2}{dx^2} + \alpha x \frac{d}{dx} .$$

Show that the Laplace transform of A_t is given by $\mathbb{E}_x \left[e^{-\beta A_t} \right] = u(t, x)$.

4. (Quadratic variation of Itō integrals). Suppose that $X : [0, \infty) \to \mathbb{R}$ is a continuous function with continuous quadratic variation $[X]$ w.r.t. a fixed sequence (π_n) of partitions such that $\mathrm{mesh}(\pi_n) \to 0$.

a) Let $F \in C^1(\mathbb{R})$. Show that the quadratic variation of $t \mapsto F(X_t)$ along (π_n) is given by

$$[F(X)]_t = \int_0^t F'(X_s)^2 \, d[X]_s.$$

b) Conclude that for $f \in C^1(\mathbb{R})$, the Itō integral $I_t = \int_0^t f(X_s) \, dX_s$ has quadratic variation

$$[I(f)]_t = \int_0^t f(X_s)^2 \, d[X]_s .$$