Institute for Applied Mathematics
Summer semester 2025
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“Introduction to Stochastic Analysis” Sheet 3

Please hand in your solutions on eCampus by Wednesday, April 30, 10 am.

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1. (Martingales and stopping times of Brownian motion).
Let $(B_t)_{t\geq 0}$ be a d-dimensional Brownian motion. Show that:

a) The following processes are martingales w.r.t. each of the filtrations $(\mathcal{F}_t^B)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^B$ denotes the right continuous filtration:

   (i) The coordinate processes $B_t^{(i)}$, $1 \leq i \leq d$,

   (ii) $B_t^{(i)} B_t^{(j)} - t \cdot \delta_{ij}$ for any $1 \leq i, j \leq d$,

   (iii) $\exp(\alpha \cdot B_t - \frac{1}{2}|\alpha|^2 t)$ for any $\alpha \in \mathbb{R}^d$.

b) If *all* sample paths $t \mapsto B_t(\omega)$ are continuous, then for every closed set $A \subset \mathbb{R}^d$, the first hitting time $T_A = \inf\{t \geq 0 : B_t \in A\}$ is a stopping time w.r.t. $(\mathcal{F}_t^B)_{t\geq 0}$.

c) For an open set $U \subset \mathbb{R}^d$, T_U is a stopping time w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ but not necessarily w.r.t. $(\mathcal{F}_t^B)_{t\geq 0}$.

2. (The Wiener integral as a martingale). Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let $g \in \mathcal{L}^2_{\text{loc}}(0, \infty)$. We consider the stochastic process $(I_t)_{t\in[0,\infty)}$ given by the Wiener integrals

$$I_t \;=\; \int_0^t g(s)\, dB_s \;:=\; \int_0^\infty 1_{[0,t]}(s) g(s)\, dB_s.$$

a) Show that the following processes are martingales w.r.t. the filtration $(\mathcal{F}_t^B)_{t\geq 0}$:

   (i) The process I_t itself,

   (ii) $I_t^2 - \int_0^t g(s)^2\, ds$,

   (iii) $\exp(\alpha I_t - \frac{1}{2}\alpha^2 \int_0^t g(s)^2\, ds)$ for any $\alpha \in \mathbb{R}$.

b) Prove that if g is bounded then for any $T > 0$, the process $(I_t)_{t\in[0,T]}$ has a modification that is Hölder continuous of order α for any $\alpha \in (0, 1/2)$.

3. (Uniform integrability).

a) For which sequences (a_n) of real numbers are the random variables

$$X_n \;= a_n \cdot I_{(0,1/n)}\ , \qquad n \in \mathbb{N},$$

uniformly integrable w.r.t the uniform distribution on the interval $(0, 1)$?

b) Show that the exponential martingale $M_t = \exp(B_t - t/2)$ of a 1-dimensional Brownian motion is not uniformly integrable.

c) Let $(M_n)_{n\in\mathbb{N}_0}$ be an $(\mathcal{F}_n)$ martingale with $\sup \mathbb{E}[\,|M_n|^p\,] < \infty$ for some $p > 1$. Prove that (M_n) converges almost surely and in L^1, and $M_n = \mathbb{E}[M_\infty \mid \mathcal{F}_n]$ for all $n \geq 0$. Hence, conclude that $|M_n - M_\infty|^p$ is uniformly integrable, and $M_n \to M_\infty$ in L^p.

4. (Stopping times). Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let S and T be $(\mathcal{F}_t)$ stopping times. Show that the following properties hold:

a) $T \wedge S$, $T \vee S$ and $T + S$ are again $(\mathcal{F}_t)$ stopping times.

b) $\mathcal{F}_T$ is a σ-algebra, and T is $\mathcal{F}_T$-measurable.

c) $S \leq T \Rightarrow \mathcal{F}_S \subseteq \mathcal{F}_T$;

d) $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$;

e) The events $\{S < T\}$, $\{S \leq T\}$ and $\{S = T\}$ are all contained in $\mathcal{F}_S \cap \mathcal{F}_T$.

5. (Simulation of Ornstein-Uhlenbeck processes I). An *Ornstein-Uhlenbeck process* is a solution of a stochastic differential equation $dX_t = -\gamma X_t\, dt + \sigma\, dB_t$, $X_0 = x_0$, i.e.,

$$X_t \;=\; x_0 \;-\; \int_0^t \gamma\, X_s\, ds \;+\; \sigma\, B_t \qquad \text{for all } t \in [0,\infty), \tag{1}$$

where $x_0 \in \mathbb{R}$ and $\gamma, \sigma \in (0,\infty)$ are given constants, and $(B_t)_{t\geq 0}$ is a Brownian motion.

a) Write down a time-discretization of (1), where $t \in h\mathbb{N}_0$ for a given step size $h > 0$.

b) Simulate a given number k of sample paths of an OU process on a time interval $[0, t_{\max}]$, and plot the result. Run the simulation for different values of h, x_0, γ and σ.

Remark on programming exercises: You can use a language of your choice. Model solutions will be provided in Python with Jupyter notebooks. Links to a brief introduction to Python and Jupyter notebooks and some examples can be found on the course homepage.