Institute for Applied Mathematics
Summer semester 2025
Andreas Eberle, Francis Lörler

UNIVERSITÄT BONN IAM

"Introduction to Stochastic Analysis" Sheet 2

Please hand in your solutions on eCampus by Wednesday, April 23, 10 am.

---

1. (Wiener integral revisited). Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. In this exercise, we give an alternative definition of the Wiener integral

$$I(g) \;=\; \int_0^\infty g(s)\, dB_s.$$

a) Suppose first that $g : [0, \infty) \to \mathbb{R}$ is a step function, i.e.,

$$g(s) \;=\; \sum_{i=0}^{n-1} a_i\, 1_{(t_i, t_{i+1}]}(s)$$

for some $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \ldots < t_n$, and $a_1, \ldots a_n \in \mathbb{R}$. Then we define

$$I(g) \;=\; \sum_{i=0}^{n-1} a_i \cdot \left(B_{t_{i+1}} - B_{t_i}\right).$$

Show that $g \mapsto I(g)$ is a linear isometry from the subspace of step functions in $L^2(0, \infty)$ to $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

b) Use this isometry to define $I(g)$ for any $g \in L^2(0, \infty)$.

c) Now suppose that $g \mapsto W(g)$ is a white noise on $[0, \infty)$. It has been shown in the lecture, that there exists a Brownian motion $(B_t)_{t\geq 0}$ such that for any $t \geq 0$, $B_t = W((0, t])$ almost surely. Show that for this Brownian motion and $g \in L^2(0, \infty)$,

$$\int_0^\infty g(s)\, dB_s \;=\; W(g) \quad \text{almost surely.}$$

2. (Brownian bridge). Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion with $B_0 = 0$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let

$$X_t \;=\; B_t - tB_1, \qquad t \in [0, 1].$$

a) Show that $(X_t)_{t\in[0,1]}$ is a centred Gaussian process and compute its covariance function $C(s, t) = \mathrm{Cov}(X_s, X_t)$.

b) Show that for $0 < t_1 < \ldots < t_n < 1$, the joint law of $X_{t_1}, \ldots, X_{t_n}$ has density

$$f(x_1, \ldots, x_n) \;=\; \sqrt{2\pi}\, \varphi_{t_1}(x_1)\, \varphi_{t_2-t_1}(x_2 - x_1) \cdots \varphi_{t_n-t_{n-1}}(x_n - x_{n-1})\, \varphi_{1-t_n}(-x_n),$$

where φ_t denotes the density of the normal distribution $N(0, t)$.

c) Explain why the law of $(X_{t_1}, \dots, X_{t_n})$ can be interpreted as the conditional law of $(B_{t_1}, \dots, B_{t_n})$ given $B_1 = 0$ (that is, $(X_t)_{t\in[0,1]}$ is a *Brownian bridge from* 0 *to* 0).

d) Verify that the two processes $(X_t)_{t\in[0,1]}$ and $(X_{1-t})_{t\in[0,1]}$ have the same distribution on $C([0,1],\mathbb{R})$ (that is, the law is *invariant under time reversal*).

3. (Brownian sheet). Suppose that $g \mapsto W(g)$ is a white noise on $[0,\infty) \times [0,\infty)$ (endowed with Lebesgue measure).

a) Show that there exists a *continuous* two-parameter process $(B_{s,t})_{s,t\in[0,\infty)}$ such that for any $s, t \geq 0$,

$$B_{s,t} \;=\; W\left((0,s] \times (0,t]\right) \qquad \text{almost surely.}$$

Verify that B is a centred two-parameter continuous Gaussian process with covariance function

$$\mathrm{Cov}\left(B_{s,t}, B_{u,v}\right) \;=\; \min(s,u) \cdot \min(t,v)\,.$$

Such a process is called a *Brownian sheet.*

b) Show that for a fixed s, the process $(B_{s,t})_{t\geq 0}$ is a multiple of a Brownian motion.

c) Prove that for any $a, b > 0$, the rescaled process $\frac{1}{ab} B_{a^2 s, b^2 t}$ is again a Brownian sheet.