Institute for Applied Mathematics
Summer semester 2025
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"Introduction to Stochastic Analysis" Sheet 0

The exercises on this page will be discussed in the tutorials during the first/second week (friday to tuesday). You do not have to submit solutions.

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1. (Revision of conditional expectations 1). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{F} \subset \mathcal{A}$ a σ-algebra, and $X : \Omega \to \mathbb{R}_+$ a non-negative random variable.

a) Define the conditional expectation $\mathbb{E}[X|\mathcal{F}]$.

b) Suppose that there exists a decomposition of Ω into disjoint sets $A_1, \dots A_n$ such that $\mathcal{F} = \sigma(\{A_1, \dots, A_n\})$. Show that

$$\mathbb{E}[X|\mathcal{F}] \;=\; \sum\nolimits_{i:\mathbb{P}[A_i]>0} \mathbb{E}[X|A_i]\, 1_{A_i}$$

is a version of the conditional expectation of X given $\mathcal{F}$.

2. (Revision of conditional expectations 2). Let $X, Y : \Omega \to \mathbb{R}_+$ be non-negative random variables. Show that $\mathbb{P}$-almost surely, the following identities hold:

a) For $\lambda \in \mathbb{R}$ we have $\mathbb{E}[\lambda X + Y|\mathcal{F}] = \lambda \mathbb{E}[X|\mathcal{F}] + \mathbb{E}[Y|\mathcal{F}]$.

b) $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]] = \mathbb{E}[X]$ and $|\mathbb{E}[X|\mathcal{F}]| \le \mathbb{E}[|X|\,|\mathcal{F}]$.

c) If $\sigma(X)$ is independent of $\mathcal{F}$, then $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$.

d) Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be measurable spaces. If $Y : \Omega \to S$ is $\mathcal{F}$-measurable, $X : \Omega \to T$ is independent of $\mathcal{F}$ and $f : S \times T \to [0, \infty)$ is product-measurable, then

$$\mathbb{E}[f(Y, X)|\mathcal{F}](\omega) = \mathbb{E}[f(Y(\omega), X)] \qquad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega.$$

3. (Revision of conditional expectations 3). Let X, Y, Z be random variables on a joint probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We define

$$\mathbb{E}[X|Y] := \mathbb{E}[X|\sigma(Y)].$$

Show the following statements:

a) If $X, Y \in \mathcal{L}^1$ are independent and identically distributed, then $\mathbb{P}$-almost surely,

$$\mathbb{E}[X|X + Y] = \frac{1}{2}(X + Y).$$

b) If Z is independent of the pair (X, Y), then $\mathbb{P}$-almost surely,

$$\mathbb{E}[X|Y, Z] = \mathbb{E}[X|Y].$$

Is this statement still true if we only assume that X and Z are independent?

“Introduction to Stochastic Analysis” Sheet 1

Please hand in your solutions on eCampus by Tuesday, April 15.

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1. (Conditional expectations). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{G} \subseteq \mathcal{A}$ a σ-algebra, and let X, Y, Z be random variables on $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose that the random variables (X, Z) and (Y, Z) have the same law (in particular X and Y have common law μ).

a) For $A \in \mathcal{A}$, consider the event $B = \{\mathbb{P}[A|\mathcal{G}] = 0\}$. Show that $B \subset A^c$ a.s.

b) Show that, if f is a non-negative function then
$$\mathbb{E}[f(X)|Z] \ = \mathbb{E}[f(Y)|Z] \ \text{ a.s.}$$

c) Let $g : \mathbb{R} \to \mathbb{R}_+$ be measurable. Suppose that $\mathbb{P}$-almost surely, $\mathbb{E}[g(Z)|X] = h_1(X)$ and $\mathbb{E}[g(Z)|Y] = h_2(Y)$. Show that $h_1 = h_2$ μ-a.s.

2. (Paley-Wiener Integral). Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$ with $B_0 = 0$. For a function $h \in C^1([0,1], \mathbb{R})$, the stochastic integral of h w.r.t. B can be defined via the integration by parts identity
$$\int_0^1 h(s) dB_s \ := \ h(1)\, B_1 - \int_0^1 h'(s)\, B_s\, ds.$$

a) Show that the random variables $\int_0^1 h(s) dB_s$ are normally distributed with mean 0 and variance $\int_0^1 h(s)^2 ds$. In particular,
$$\mathbb{E}\left[\left(\int_0^1 h(s) dB_s\right)^2\right] = \int_0^1 h(s)^2 ds.$$

b) Use this isometry to define the integral $\int_0^1 h(s) dB_s$ for an arbitrary $h \in L^2(0,1)$.

c) Compute the covariance of two integrals $\int_0^1 g(s) dB_s$ and $\int_0^1 h(s) dB_s$ with $g, h \in L^2(0,1)$. What do you obtain for $g = 1_A$ and $h = 1_B$?

3. (Gaussian martingales). A process $(M_n)_{n=0,1,2,\ldots}$ is called Gaussian if for every n, the vector $(M_0, \ldots, M_n)$ is normally distributed. Let (M_n) be a Gaussian martingale.

a) Show that (M_n) has independent increments, i.e., the random variable $M_{n+1} - M_n$ is independent of the σ-algebra $\mathcal{F}_n = \sigma(M_0, \ldots, M_n)$.

b) We set $\sigma_0^2 = \mathrm{Var}(M_0)$ and $\sigma_k^2 = \mathrm{Var}(M_k - M_{k-1})$ for $k \geq 1$. Compute the predictable increasing process $\langle M \rangle_n$ in the Doob decomposition of the submartingale (M_n^2).

c) Show that for every $\theta \in \mathbb{R}$, $Z_n^\theta = e^{\theta M_n - \frac{1}{2}\theta^2 \langle M \rangle_n}$ is a martingale. Does it converge a.s.?

4. (Martingales, supermartingales and stopping times).

a) Let (X_n) be a supermartingale s.t. $\mathbb{E}[X_n]$ is constant. Show that (X_n) is a martingale.

b) Let (X_n) be an integrable process adapted to the filtration $(\mathcal{F}_n)$. Show that (X_n) is a martingale if and only if $\mathbb{E}[X_T] = \mathbb{E}[X_0]$ for every bounded $(\mathcal{F}_n)$ stopping time T.