FLUCTUATIONS OF THE FREE ENERGY IN P-SPIN SK MODELS ON TWO SCALES

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ABSTRACT. 20 years ago, Bovier, Kurkova, and Löwe [5] proved a central limit theorem (CLT) for the fluctuations of the free energy in the p-spin version of the Sherrington-Kirkpatrick model of spin glasses at high temperatures. In this paper we improve their results in two ways. First, we extend the range of temperatures to cover the entire regime where the quenched and annealed free energies are known to coincide. Second, we identify the main source of the fluctuations as a purely coupling dependent term, and we show a further CLT for the deviation of the free energy around this random object.

1. Introduction

The p-spin interaction version of the Sherrington-Kirkpatrick [12] is a spin system defined on the hypercube $S_N \equiv \{-1,+1\}^N$ where the random Hamiltonian is given in terms of a Gaussian process $X_\cdot : S_N \to \mathbb{R}$ given by

$$X_\sigma = \binom{N}{p}^{-\frac{1}{2}} \sum_{1\leq i_1<i_2,\dots<i_p\leq N} J_{i_1,i_2,\dots,i_p}\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_p}, \tag{1.1}$$

where the $\{J_{i_1,\dots,i_p}\}_{i_1,\dots,i_p=1}^{\infty}$ is a family of independent, standard normal random variables defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. Alternatively, X is characterised uniquely as the Gaussian field on S_N withe mean zero and covariance

$$\mathbb{E}\left(X_\sigma X_{\sigma'}\right) \equiv f_{p,N}\left(R_N(\sigma,\sigma')\right), \tag{1.2}$$

where

$$R_N(\sigma,\sigma') \equiv \frac{1}{N}(\sigma,\sigma') \equiv \frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma'_i \tag{1.3}$$

is the *overlap* between the configurations σ,σ', and $f_{p,N}$ is of the form (see [5])

$$f_{p,N}(x) = \sum_{k=0}^{[p/2]} d_{p-2k}N^{-k}x^{p-2k}(1+O(1/N)), \tag{1.4}$$

where

$$d_{p-2k} \equiv (-1)^k\binom{p}{2k}k!!. \tag{1.5}$$

In particular,

$$f_{p,N}(x) = x^p\left(1+O(1/N)\right), \quad \text{uniformly for } x\in[-1,1]. \tag{1.6}$$

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The model with $p = 2$ is the classical SK model, introduced in [12], and the general version with $p > 3$, by Gardner [9]. The *Hamiltonian* is given by

$$H_N(\sigma) \equiv -\sqrt{N} X_\sigma, \tag{1.7}$$

and the *partition function* is

$$Z_N(\beta) \equiv \mathbb{E}_\sigma\left[\mathrm{e}^{-\beta H_N(\sigma)}\right] \equiv 2^{-N} \sum_{\sigma \in S_N} \mathrm{e}^{\beta\sqrt{N} X_\sigma}. \tag{1.8}$$

Finally, minus the *free energy* is

$$F_N(\beta) \equiv \frac{1}{N} \ln Z_N(\beta). \tag{1.9}$$

For $m \in (0, 1)$, let

$$\phi(m) \equiv \frac{1-m}{2} \ln(1-m) + \frac{1+m}{2} \ln(1+m), \tag{1.10}$$

and

$$\beta_p^2 \equiv \inf_{0<m<1} (1 + m^{-p})\phi(m), \tag{1.11}$$

for $p \geq 3$, and $\beta_2 = 1$. It is a well-known consequence of Gaussian concentration of measure theorems, that the free energy is self-averaging in the sense that

$$\lim_{N\uparrow\infty} F_N(\beta) = \lim_{N\uparrow\infty} \mathbb{E}\left[F_N(\beta)\right], \text{a.s.}. \tag{1.12}$$

The existence of the limit on the right-hand side was established in a celebrated paper by Guerra and Toninelli [10]. For $\beta < \beta_p$, it is even true that the so-called *quenched free energy* on the right-hand side is equal to the so-called *annealed free energy*, that is

$$\lim_{N\uparrow\infty} \mathbb{E}\left[F_N(\beta)\right] = \lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}[Z_N(\beta)] = \frac{\beta^2}{2}. \tag{1.13}$$

This fact was first proven for $p = 2$ by Aizenman, Lebowitz, and Ruelle [1] and a very simple proof was given later by Talagrand [13]. The proof in the case $p \geq 3$ is also due to Talagrand [14]. Note that

$$\lim_{p\uparrow+\infty} \beta_p = \sqrt{2 \ln 2}, \tag{1.14}$$

which is the well-known critical temperature of the REM [8]. It is, however, not known whether β_p is the true critical value in general. It is natural to ask about fluctuations around this limit. This was first done by Comets and Neveu [7], who used the martingale central limit theorem (CLT) for the free energy in the case $p = 2$ for all $\beta < 1$. The case $p \geq 3$ was analysed by Bovier, Kurkova, and Löwe [5], also using martingale methods. They established a CLT in a range $\beta < \tilde{\beta}_p$, for some $\tilde{\beta}_p < \beta_p$. Our first result extends this to the entire range $\beta < \beta_p$.

Theorem 1.1. *For all $p \geq 3$ and $\beta < \beta_p$,*

$$N^{\frac{p}{2}} \left(F_N(\beta) - \frac{\beta^2}{2}\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{\beta^4 p!}{2}\right), \text{as } N \uparrow \infty. \tag{1.15}$$

The proof of Theorem 1.1 is very different from that in [5] and in a sense closer to that of Aizenman et al. [1] in the case $p = 2$. In fact, we show that the limiting Gaussian comes from a very explicit term

$$J_N(\beta) \equiv \frac{1}{2N} \mathbb{E}_\sigma\left(\beta^2 H_N(\sigma)^2\right) = \frac{\beta^2}{2\binom{N}{p}} \sum_{1\leq i_1<\ldots<i_p\leq N} J^2_{i_1,i_2,\ldots,i_p}. \tag{1.16}$$

$J_N(\beta)$ is a sum of independent square integrable random variables, and hence by the law of large numbers, for all β,

$$\lim_{N\to\infty} J_N(\beta) = \frac{\beta^2}{2}\,, \text{a.s.}, \tag{1.17}$$

and by the central limit theorem,

$$N^{\frac{p}{2}}\left(J_N(\beta) - \frac{\beta^2}{2}\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{\beta^4 p!}{2}\right), \text{as } N \uparrow \infty. \tag{1.18}$$

That J_N and the F_N have the same limits is not a coincidence. In fact, we prove Theorem 1.1 by proving that

$$\lim_{N\uparrow\infty} N^{\frac{p}{2}}\left(F_N(\beta) - J_N(\beta)\right) = 0, \tag{1.19}$$

in probability. This naturally leads to the question whether upon proper rescaling, the quantity $F_N(\beta) - J_N(\beta)$ converges to a random variable. The positive answer is the main result of this paper and given by the following theorem.

Theorem 1.2. *For $p > 2$ and for all $\beta < \beta_p$, we have*

$$A_N(p)\Big(F_N(\beta) - J_N(\beta)\Big) \xrightarrow{\mathcal{D}} \mathcal{N}\left(\mu(\beta, p), \sigma(\beta, p)^2\right), \tag{1.20}$$

where

(i) *For p even,*

$$A_N(p) = N^{\left(\frac{3p}{4} - \frac{1}{2}\right)}, \quad \mu(\beta, p) = 0, \tag{1.21}$$

and

$$\sigma(\beta, p)^2 = \frac{\beta^6}{3}\mathbb{E}\left[\left(\sum_{k=0}^{p/2} d_{p-2k} X^{p-2k}\right)^3\right]. \tag{1.22}$$

(ii) *For p odd,*

$$A_N(p) = N^{p-1}, \quad \mu(\beta, p) = \frac{-\beta^4 p!}{4}, \tag{1.23}$$

and

$$\sigma(\beta, p)^2 = \frac{\beta^8}{12}\mathbb{E}\left[\left(\sum_{k=0}^{[p/2]} d_{p-2k} X^{p-2k}\right)^4\right] - \frac{\beta^8 p!^2}{8}. \tag{1.24}$$

Here X is a standard normal random variable and $d_{p-2k} = (-1)^k \frac{p(p-1)\dots(p-2k+1)}{2^k k!}$.

Compared to (1.15), Theorem 1.2 provides a higher-level resolution of the limiting picture. In fact, in the course of the proof we also identify exactly the terms arising in the expansion of the partition function that converge to the Gaussian in (1.20). Thus, one might envision that, once these terms are again subtracted, on a smaller scale, there appears yet another limit theorem. This might even continue ad infinitum. To prove such a result appears, however, rather formidable and will be left to future research.

It is interesting to compare this picture with the $p = 2$ case. In that case, the variance of the limiting Gaussian distribution blows up at the critical temperature, and thus detects the phase transition. For $p > 2$, this is not the case for the Gaussian from Theorem 1.1, nor for the corrections given by Theorem 1.2. This is of course completely in line with the predictions by theoretical physics pertaining to the so-called Gardner's transition [9].

Results similar to Theorem 1.1 have been obtained for several related models. Chen et al. [6] obtained analogous results to [5] for mixed p-SK models, i.e. where the Hamiltonian is given as a linear combination of terms of type (1.1) with different p where only even p appear, and recently this was extended to the general case by Banerjee and Belius [3]. For spherical SK-models, related results were obtained by Baik and Lee [2]. We are not aware of any results like Theorem 1.2. The paper is organised as follows. In the next section, we present the proof of Theorem 1.1. Many of the results obtained in the course of the proof are re-used in Section 3 where Theorem 1.2 is proven. In the appendix we state two frequently used facts about Gaussian random variables for quick reference.

2. **Proof of Theorem 1.1.**

In view of (1.18), to prove Theorem 1.1, it is enough to establish that (1.19) holds for all $\beta < \beta_p$. Setting

$$\mathcal{Z}_N(\beta) \equiv Z_N(\beta)\mathrm{e}^{-NJ_N(\beta)}, \tag{2.1}$$

this amounts to showing that, for $\beta < \beta_p$,

$$\lim_{N\to\infty} N^{\frac{p-2}{2}} \ln \mathcal{Z}_N(\beta) = 0, \text{ in probability.} \tag{2.2}$$

The proof of (2.2) turns out to be remarkably difficult if the entire range $\beta < \beta_p$ is to be covered. This will require a truncation. For $\epsilon > 0$, we set

$$\mathcal{Z}_N(\beta) = Z_\epsilon^{\leq} + Z_\epsilon^{>}, \tag{2.3}$$

where

$$Z_\epsilon^{\leq} \equiv \mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\mathbb{1}_{\{|-H_N(\sigma)-\beta N|\leq\epsilon\beta N\}}\right)\mathrm{e}^{-NJ_N(\beta)}, \quad Z_\epsilon^{>} \equiv \mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\mathbb{1}_{\{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\right)\mathrm{e}^{-NJ_N(\beta)}, \tag{2.4}$$

where we dropped obvious dependencies on the parameters β, N to lighten the notation.

We decompose

$$N^{\frac{p-2}{2}} \ln \mathcal{Z}_N(\beta) = N^{\frac{p-2}{2}} \ln\left(\frac{\mathcal{Z}_N(\beta)}{Z_\epsilon^{\leq}}\right) + N^{\frac{p-2}{2}} \ln\left(\frac{Z_\epsilon^{\leq}}{\mathbb{E}[Z_\epsilon^{\leq}]}\right) + N^{\frac{p-2}{2}} \ln \mathbb{E}[Z_\epsilon^{\leq}]. \tag{2.5}$$

The assertion of the theorem then follows from the fact that all three terms on the right-hand side of (2.5) converge to zero in probability.

Proposition 2.1. (i) *For any $q \in \mathbb{N}$, $\beta < \beta_p$ and small enough $\epsilon = \epsilon(\beta, p) > 0$,*

$$\lim_{N\uparrow+\infty} N^q \ln\left(\frac{\mathcal{Z}_N(\beta)}{Z_\epsilon^{\leq}}\right) = 0, \textit{ in probability.} \tag{2.6}$$

(ii) *For $\beta < \beta_p$ and small enough $\epsilon = \epsilon(\beta, p) > 0$,*

$$\lim_{N\uparrow+\infty} N^{\frac{p-2}{2}} \ln\left(\frac{Z_\epsilon^{\leq}}{\mathbb{E}Z_\epsilon^{\leq}}\right) = 0, \textit{ in probability,} \tag{2.7}$$

(iii) *For any $\beta, \epsilon > 0$,*

$$\lim_{N\uparrow+\infty} N^{\frac{p-2}{2}} \ln \mathbb{E}[Z_\epsilon^{\leq}] = 0\,. \tag{2.8}$$

Remark. The fact that (2.6) holds for all $q \in \mathbb{N}$ is not needed here, but will be used in the proof of Theorem 1.2.

The proof of Proposition 2.1 relies on computations of moments that are combinatorially rather complex.

We introduce some convenient notation. First, we denote by I_N the set of all strictly increasing p-tupels in $\{1, \ldots, N\}$,

$$I_N \equiv \left\{(i_1, i_2, \ldots, i_p) \in \{1, \ldots, N\}^p,\ i_1 < i_2 < \cdots < i_p\right\}. \tag{2.9}$$

For $A = (i_1, \dots, i_p) \in I_N$ we write

$$\sigma_A \equiv \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_p}, \text{and} \quad J_A \equiv J_{i_1,\dots,i_p}. \tag{2.10}$$

We abbreviate

$$a_N \equiv \sqrt{N}\binom{N}{p}^{-1/2}. \tag{2.11}$$

We can thus write

$$H_N(\sigma) = -a_N \sum_{A\in I_N} J_A\sigma_A, \qquad \text{and} \quad J_N(\beta) = \frac{\beta^2}{2N}a_N^2 \sum_{A\in I_N} J_A^2. \tag{2.12}$$

For $a_N, b_N \geq 0$ we write $a_N \lesssim b_N$ if $a_N \leq Cb_N$ for some numerical constant $C > 0$.

Finally, we will denote by c > 0 a numerical constant, not necessarily the same at different occurrences.

2.1. **First moments of $\mathcal{Z}_N(\beta)$ and $Z_\epsilon^\leq$, and proof of part (iii) of Proposition (2.1).** We will show that

Lemma 2.2. *With the notation above,*

$$\mathbb{E}\left[\mathcal{Z}_N(\beta)\right] = 1 - \frac{\beta^4}{4}Na_N^2 + \frac{\beta^8}{32}N^2a_N^4 + O\left(N^{3-2p}\right). \tag{2.13}$$

Proof. Interchanging the order of integration, we have

$$\mathbb{E}\left[\mathcal{Z}_N(\beta)\right] = \mathbb{E}\left[\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)-NJ_N(\beta)}\right)\right] = \mathbb{E}_\sigma\left(\mathbb{E}\left[\mathrm{e}^{-\beta H_N(\sigma)-NJ_N(\beta)}\right]\right). \tag{2.14}$$

Using (2.12) and the independence to the J_A,

$$\mathbb{E}\left[\mathrm{e}^{-\beta H_N(\sigma)-NJ_N(\beta)}\right] = \prod_{A\in I_N}\mathbb{E}\left[\mathrm{e}^{\beta a_N J_A\sigma_A - \frac{\beta^2}{2}a_N^2J_A^2}\right]. \tag{2.15}$$

Computing the Gaussian integral, we get

$$\mathbb{E}\left[\mathrm{e}^{\beta a_N J_A\sigma_A - \frac{\beta^2}{2}a_N^2J_A^2}\right] = \mathrm{e}^{\left(\frac{\beta^2a_N^2\sigma_A^2}{2\left(1+\beta^2a_N^2\right)}\right)}\frac{1}{\sqrt{1+\beta^2a_N^2}}. \tag{2.16}$$

Since $\sigma_A^2 = 1$, this implies that

$$\mathbb{E}\left[\mathcal{Z}_N(\beta)\right] = \exp\left(|I_N|\left(\frac{\beta^2a_N^2}{2\left(1+\beta^2a_N^2\right)} - \frac{1}{2}\ln\left(1+\beta^2a_N^2\right)\right)\right). \tag{2.17}$$

Moreover, using that $|I_N| = \binom{N}{p}$, and by Taylor expansion we obtain

$$\begin{aligned}\mathbb{E}\left[\mathcal{Z}_N(\beta)\right] &= \exp\left(\frac{\binom{N}{p}}{2}\left(\beta^2a_N^2 - \beta^4a_N^4 + O(a_N^6) - \beta^2a_N^2 + \frac{\beta^4a_N^4}{2}\right)\right) \\ &= 1 - \frac{\beta^4}{4}Na_N^2 + \frac{\beta^8}{32}N^2a_N^4 + O\left(N^{3-2p}\right),\end{aligned} \tag{2.18}$$

which is (2.13). □

From (2.13) it follows that $\ln\mathbb{E}\left[\mathcal{Z}_N(\beta)\right] = O(Na_N^2)$, and $Na_N^2 = O(N^{2-p})$, it follows that $N^{(p-2)/2}\ln\mathbb{E}\left[\mathcal{Z}_N(\beta)\right] = O(N^{1-p/2})$, which tends to zero for $p \geq 3$. The next lemma states that $Z_\epsilon^\leq$ and $\mathcal{Z}_N(\beta)$ are exponentially close, which will imply (2.8),

Lemma 2.3. *For any $\epsilon > 0$,*

$$\mathbb{E}\left[\left|Z_\epsilon^\le - \mathcal{Z}_N(\beta)\right|\right] \le \exp\left(-\beta^2 N\epsilon^2/2 + O(N^{2-p})\right). \tag{2.19}$$

Proof. Since $\mathcal{Z}_N(\beta) - Z_\epsilon^\le = Z_\epsilon^>$, we just have to control the expectation of the latter. Interchanging the order of integration, we obtain, using the Hölder inequality,

$$\begin{aligned}\mathbb{E}\left(Z_\epsilon^>\right) &= \mathbb{E}_\sigma\left(\mathbb{E}\left(\mathrm{e}^{-\beta H_N(\sigma)}\mathbb{1}_{\{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\mathrm{e}^{-NJ_N(\beta)}\right)\right)\\ &\le \mathbb{E}_\sigma\left(\mathbb{E}\left(\mathrm{e}^{-q_1\beta H_N(\sigma)}\mathbb{1}_{\{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\right)^{1/q_1}\mathbb{E}\left(\mathrm{e}^{-q_2NJ_N(\beta)}\right)^{1/q_2}\right)\\ &= \mathbb{E}_\sigma\left(\mathbb{E}\left(\mathrm{e}^{q_1\beta\sqrt{N}X_\sigma}\mathbb{1}_{\{|X_\sigma-\beta\sqrt{N}|>\epsilon\beta\sqrt{N}\}}\right)^{1/q_1}\mathbb{E}\left(\mathrm{e}^{-q_2NJ_N(\beta)}\right)^{1/q_2}\right),\end{aligned} \tag{2.20}$$

for $1/q_1 + 1/q_2 = 1$. Classical Gaussian estimates (see **Fact I** in the Appendix) yield that

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{q_1\beta\sqrt{N}X_\sigma}\mathbb{1}_{\{|X_\sigma-\beta\sqrt{N}|>\epsilon\beta\sqrt{N}\}}\right) &= \mathbb{E}\left(\mathrm{e}^{q_1\beta\sqrt{N}X_\sigma}\mathbb{1}_{\{X_\sigma>(1+\epsilon)\beta\sqrt{N}\}}\right)+\mathbb{E}\left(\mathrm{e}^{q_1\beta\sqrt{N}X_\sigma}\mathbb{1}_{\{X_\sigma<(1-\epsilon)\beta\sqrt{N}\}}\right)\\ &\le \max\nolimits_{z\in\{-1,1\}}\mathrm{e}^{-\frac{(1+z\epsilon)^2\beta^2N}{2}+q_1(1+z\epsilon)\beta^2N}\\ &= \mathrm{e}^{-\frac{\beta^2N}{2}\left(-q_1+\epsilon^2+(1-q_1)(1+2\epsilon)\right)}\end{aligned} \tag{2.21}$$

for N large enough and $q_1 < 1+\epsilon$. Note that this bound is independent of σ. It remains to calculate the second term on the r.h.s. of (2.20). By independence of the J's,

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{-q_2NJ_N(\beta)}\right) &= \left[\mathbb{E}\left(\mathrm{e}^{-\frac{q_2\beta^2}{2}a_N^2J_A^2}\right)\right]^{\binom{N}{p}} = \left(1+q_2\beta^2a_N^2\right)^{-\frac{1}{2}\binom{N}{p}}\\ &= \exp\left(-\frac{\binom{N}{p}}{2}\ln\left(1+\frac{N\beta^2q_2}{\binom{N}{p}}\right)\right) = \exp\left(-\frac{N\beta^2q_2}{2}+O(N^{2-p})\right).\end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22), we obtain, for any $1+\epsilon > q_1 > 1$,

$$\mathbb{E}\left(Z_\epsilon^>\right) \le \exp\left(-\frac{\beta^2N}{2q_1}\left(\epsilon^2+(1-q_1)(1+2\epsilon)-O(N^{1-p})\right)\right). \tag{2.23}$$

But this implies the assertion of the lemma. □

Note that Lemma 2.3 and (2.18) imply that

$$\mathbb{E}\left[Z_\epsilon^\le\right] = 1-\frac{\beta^4}{4}Na_N^2+\frac{\beta^8}{32}N^2a_N^4+O\left(N^{3-2p}\right). \tag{2.24}$$

Combining Lemma 2.2 and Lemma 2.3 proves (2.8).

2.2. The second moment of $Z_\epsilon^\le$, and proof of part (ii) of Proposition 2.1.

We set

$$\Xi_\epsilon \equiv \frac{Z_\epsilon^\le - \mathbb{E}[Z_\epsilon^\le]}{\mathbb{E}[Z_\epsilon^\le]}. \tag{2.25}$$

(2.7) is then equivalent to the following lemma.

Lemma 2.4. *For any $\varepsilon > 0$ and $\beta < \beta_p$,*

$$\lim_{N\uparrow\infty}\mathbb{P}\left(\left|N^{\frac{p-2}{2}}\ln(1+\Xi_\epsilon)\right|\ge\varepsilon\right)=0. \tag{2.26}$$

Proof. Using the Chebyshev inequality and the fact that, for $|x| \le 1/10$, $(\mathrm{e}^x-1)^2 \ge x^2/2$, for N large enough,

$$\mathbb{P}\left(\left|N^{\frac{p-2}{2}}\ln(1+\Xi_\epsilon)\right|\ge\varepsilon\right)\le\frac{\mathbb{E}\left[\Xi_\epsilon^2\right]}{\left(\mathrm{e}^{\varepsilon N^{1-p/2}}-1\right)^2}+\frac{\mathbb{E}\left[\Xi_\epsilon^2\right]}{\left(\mathrm{e}^{-\varepsilon N^{1-p/2}}-1\right)^2}\le 8\varepsilon^{-2}N^{p-2}\mathbb{E}\left[\Xi_\epsilon^2\right]. \tag{2.27}$$

Since $\mathbb{E}\left[\Xi_\epsilon^2\right] = \frac{\mathbb{E}[(Z_\epsilon^{\leq})^2]-\mathbb{E}[Z_\epsilon^{\leq}]^2}{(\mathbb{E}[Z_\epsilon^{\leq}])^2}$, and $\mathbb{E}\left[Z_\epsilon^{\leq}\right]$ is already computed, we only need to compute a precise bound on the second moment of $Z_\epsilon^{\leq}$.

We write $\mathbb{E}_{\sigma,\sigma'} = \mathbb{E}_\sigma \mathbb{E}_{\sigma'}$ and set $\Gamma_N \equiv \{-1, -1-\frac{2}{N}, \dots, 1\}$. Then, for any function $G:\mathbb{R}\to\mathbb{R}$, one has

$$\begin{aligned}\mathbb{E}_{\sigma,\sigma'}\left[G\left(\binom{N}{p}^{-1}\sum_{i_1<i_2<\dots<i_p}\sigma_{i_1}\sigma'_{i_1}\dots\sigma_{i_p}\sigma'_{i_p}\right)\right] &= \mathbb{E}_{\sigma,\sigma'}\left[G\left(\mathrm{Cov}(X_\sigma, X_{\sigma'})\right)\right]\\ &= \sum_{m\in\Gamma_N} G[f_{p,N}(m)]p_N(m), \qquad (2.28)\end{aligned}$$

where $p_N(m) \equiv \mathbb{P}_{\sigma,\sigma'}(R_N(\sigma,\sigma')=m)$.

With this in mind, we split the second moment according to the value of the overlap

$$\begin{aligned}\mathbb{E}\left[\left(Z_\epsilon^{\leq}\right)^2\right] &= \mathbb{E}_{\sigma,\sigma'}\mathbb{E}\left(e^{\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathbb{1}_{\{|-H_N(\sigma)-\beta N|\leq\epsilon\beta N\}}\mathbb{1}_{\{|-H_N(\sigma')-\beta N|\leq\epsilon\beta N\}}e^{-2NJ_N(\beta)}\right)\\ &= \mathbb{E}_{\sigma,\sigma'}\mathbb{E}\left(e^{\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathbb{1}_{\{|-H_N(\sigma)-\beta N|\leq\epsilon\beta N\}}\mathbb{1}_{\{|-H_N(\sigma')-\beta N|\leq\epsilon\beta N\}}e^{-2NJ_N(\beta)}\mathbb{1}_{\{|R_N(\sigma,\sigma')|<\delta\}}\right)\\ &\quad+\mathbb{E}_{\sigma,\sigma'}\mathbb{E}\left(e^{\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathbb{1}_{\{|X_\sigma-\beta\sqrt{N}|\leq\epsilon\beta\sqrt{N}\}}\mathbb{1}_{\{|X_{\sigma'}-\beta\sqrt{N}|\leq\epsilon\beta\sqrt{N}\}}e^{-2NJ_N(\beta)}\mathbb{1}_{\{|R_N(\sigma,\sigma')|\geq\delta\}}\right)\\ &\quad\equiv A+B, \qquad (2.29)\end{aligned}$$

where $2\epsilon < \delta^p$ and $\delta^{p-2} < \frac{1}{2\beta_p^2}$. We will now prove that the B-term (large overlap) is subexponentially small and compute the leading orders of the A-term.

Lemma 2.5. *For all $\beta<\beta_p$, there exists $\epsilon_0>0$ and a constant $\mathfrak{c}$ such that, for all $0\leq\epsilon<\epsilon_0$,*

$$B \leq \exp\left(-\mathfrak{c}N\right). \qquad (2.30)$$

Lemma 2.6. *For any β,*

$$A = 1 - \frac{\beta^4 N a_N^2}{2} + O\left(N^{3-3p/2}\right). \qquad (2.31)$$

Proof of Lemma 2.5. To simplify the notation, set $B_N \equiv \{|R_N(\sigma,\sigma')|\geq\delta\}$. We simplify the constraints by using that

$$\mathbb{1}_{\{|X_\sigma-\beta\sqrt{N}|\leq\epsilon\beta\sqrt{N}\}}\mathbb{1}_{\{|X_{\sigma'}-\beta\sqrt{N}|\leq\epsilon\beta\sqrt{N}\}} \leq \mathbb{1}_{\{|X_\sigma+X_{\sigma'}-2\beta\sqrt{N}|\leq2\epsilon\beta\sqrt{N}\}}. \qquad (2.32)$$

By Hölder's inequality, we then get

$$B \leq \mathbb{E}_{\sigma,\sigma'}\left(\mathbb{E}\left(e^{q_1\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathbb{1}_{\{|X_\sigma+X_{\sigma'}-2\beta\sqrt{N}|\leq2\epsilon\beta\sqrt{N}\}}\right)^{\frac{1}{q_1}}\mathbb{E}\left(e^{-2q_2NJ_N(\beta)}\right)^{\frac{1}{q_2}}\mathbb{1}_{B_N}\right), \qquad (2.33)$$

with $q_1,q_2\geq1$ satisfying $1/q_1+1/q_2=1$. Since $X_\sigma+X_{\sigma'}$ is a Gaussian random variable with mean zero and variance $2(1+f_{p,N}(R_N(\sigma,\sigma'))$, the right hand side can be written as

$$\mathbb{E}_{\sigma,\sigma'}\left(\mathbb{E}\left(e^{q_1\beta\sqrt{N\left(2+2f_{p,N}(R_N(\sigma,\sigma'))\right)}\xi}\mathbb{1}_{\left\{\left|\xi\sqrt{2\left(1+f_{p,N}(R_N(\sigma,\sigma'))\right)}-2\beta\sqrt{N}\right|\leq2\epsilon\beta\sqrt{N}\right\}}\right)^{\frac{1}{q_1}}\mathbb{E}\left(e^{-2q_2NJ_N(\beta)}\right)^{\frac{1}{q_2}}\mathbb{1}_{B_N}\right) \qquad (2.34)$$

where ξ is a standard Gaussian. As in (2.22),

$$\mathbb{E}\left(e^{-2q_2NJ_N(\beta)}\right)^{\frac{1}{q_2}} \leq e^{-\beta^2N+O\left(N^{2-p}\right)}, \qquad (2.35)$$

and for the first term, we use the following decomposition

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{e}^{q_1\beta\sqrt{N\left(2+2f_{p,N}(R_N(\sigma,\sigma'))\right)}\xi}\mathbb{1}_{\left\{\left|\xi\sqrt{2\left(1+f_{p,N}(R_N(\sigma,\sigma'))\right)}-2\beta\sqrt{N}\right|\le 2\epsilon\beta\sqrt{N}\right\}}\right)\left(\mathbb{1}_{f_{p,N}(R_N(\sigma,\sigma'))\le 0}+\mathbb{1}_{f_{p,N}(R_N(\sigma,\sigma'))>0}\right)\\
&\le \mathbb{E}\left(\mathrm{e}^{q_1\beta\sqrt{N\left(2+2f_{p,N}(R_N(\sigma,\sigma'))\right)}\xi}\mathbb{1}_{\left\{\frac{2\beta\sqrt{N}(1-\epsilon)}{\sqrt{2\left(1+f_{p,N}(R_N(\sigma,\sigma'))\right)}}\le\xi\right\}}\right)\mathbb{1}_{f_{p,N}(R_N(\sigma,\sigma'))\le 0}\\
&+\mathbb{E}\left(\mathrm{e}^{q_1\beta\sqrt{N\left(2+2f_{p,N}(R_N(\sigma,\sigma'))\right)}\xi}\mathbb{1}_{\left\{\xi\le\frac{2\beta\sqrt{N}(1+\epsilon)}{\sqrt{2\left(1+f_{p,N}(R_N(\sigma,\sigma'))\right)}}\right\}}\right)\mathbb{1}_{f_{p,N}(R_N(\sigma,\sigma'))>0},
\end{aligned}\tag{2.36}$$

where we use the fact that

$$\mathbb{1}_{\left\{\left|\xi\sqrt{2\left(1+f_{p,N}(R_N(\sigma,\sigma'))\right)}-2\beta\sqrt{N}\right|\le 2\epsilon\beta\sqrt{N}\right\}}=\mathbb{1}_{\left\{\frac{2\beta\sqrt{N}(1-\epsilon)}{\sqrt{2\left(1+f_{p,N}(R_N(\sigma,\sigma'))\right)}}\le\xi\right\}}\mathbb{1}_{\left\{\xi\le\frac{2\beta\sqrt{N}(1+\epsilon)}{\sqrt{2\left(1+f_{p,N}(R_N(\sigma,\sigma'))\right)}}\right\}}$$

for the second line and by estimating one of the indicator function by 1 in both cases. On B_N, we can now use the first Classical Gaussian estimate of the Appendix (see **Fact I** in the Appendix) for the term in the second line of (2.36) because

$$\frac{2\beta\sqrt{N}(1-\epsilon)}{\sqrt{2\left(1+f_{p,N}\left(R_N(\sigma,\sigma')\right)\right)}}>q_1\beta\sqrt{N\left(2+2f_{p,N}\left(R_N(\sigma,\sigma')\right)\right)},$$

for q_1 small enough. On B_N, we can use the second Classical Gaussian estimate of the Appendix (see **Fact I** in the Appendix) for the term in the third line of (2.36). The two Gaussian estimates yield

$$\mathbb{E}\left(\mathrm{e}^{q_1\beta\sqrt{N\left(2+2f_{p,N}(R_N(\sigma,\sigma'))\right)}\xi}\mathbb{1}_{\left\{\left|\xi\sqrt{2\left(1+f_{p,N}(R_N(\sigma,\sigma'))\right)}-2\beta\sqrt{N}\right|\le 2\epsilon\beta\sqrt{N}\right\}}\right)^{\frac{1}{q_1}}\mathbb{1}_{B_N}\tag{2.37}$$

$$\le 2^{\frac{1}{q_1}}\max_{z\in\{-1,1\}}\exp\left(-\frac{(1+z\epsilon)^2\beta^2N}{q_1\left(1+f_{p,N}\left(R_N(\sigma,\sigma')\right)\right)}+(1+z\epsilon)2\beta^2N\right)\mathbb{1}_{B_N}.\tag{2.38}$$

Combining these two steps, we obtain

$$B\le 2^{\frac{1}{q_1}}\mathbb{E}_{\sigma,\sigma'}\left(\max_{z\in\{-1,1\}}\mathbb{1}_{B_N}\exp\left(-\frac{(1+z\epsilon)^2\beta^2N}{q_1\left(1+f_{p,N}\left(R_N(\sigma,\sigma')\right)\right)}+(1+z\epsilon)2\beta^2N-\beta^2N+O\left(N^{2-p}\right)\right)\right).\tag{2.39}$$

Since this holds for all $q_1>1$, one sees that the exponential term in (2.39) is bounded by

$$\max_{z\in\{-1,1\}}\exp\left(-\beta^2N\left(\frac{\epsilon^2-(1+2\epsilon z)f_{p,N}\left(R_N(\sigma,\sigma')\right)}{\left(1+f_{p,N}\left(R_N(\sigma,\sigma')\right)\right)}+O\left(N^{1-p}\right)\right)\right).\tag{2.40}$$

By Stirlings estimate, we have

$$p_N(m)=\binom{N}{N\frac{(1+m)}{2}}2^{-N}\le\exp\left(-N\phi(m)\right).\tag{2.41}$$

Using (2.28) and plugging (2.40) and (2.41) into (2.39) gives

$$B\lesssim\sum_{\substack{m\in\Gamma_N,\\|m|\ge\delta}}\max_{z\in\{-1,1\}}\exp\left(N\left(-\beta^2\frac{\epsilon^2-(1+2z\epsilon)f_{p,N}(m)}{\left(1+f_{p,N}(m)\right)}-\phi(m)\right)\right).\tag{2.42}$$

We write

$$\begin{aligned}\delta_N &\equiv -\beta^2 N \frac{\epsilon^2-(1+2z\epsilon)f_{p,N}(m)}{\left(1+f_{p,N}(m)\right)} - \phi(m)N \\ &= -\beta^2 N \frac{\epsilon^2}{1+f_{p,N}(m)} + \frac{f_{p,N}(m)N}{1+f_{p,N}(m)}\left[\beta^2(2z\epsilon+1)-\left(1+f_{p,N}(m)^{-1}\right)\phi(m)\right] \\ &\leq \frac{f_{p,N}(m)N}{1+f_{p,N}(m)}\left[\beta^2(2z\epsilon+1)-\left(1+f_{p,N}(m)^{-1}\right)\phi(m)\right]. \end{aligned} \tag{2.43}$$

If p is even or $m \geq 0$, recalling that $\beta_p^2 \equiv \inf_{0<m<1}(1+m^{-p})\phi(m)$, the last line in (2.43) is

$$\leq \frac{f_{p,N}(m)N}{1+f_{p,N}(m)}\left[\beta^2(2\epsilon+1)-\beta_p^2\right].$$

Since $f_{p,N}(m) = m^p + O(1/N)$, for $|m| \geq \delta$,

$$\delta_N \leq -\frac{N\delta^p + O(1)}{2}(\beta_p^2 - \beta^2(1+2\epsilon)). \tag{2.44}$$

This gives

$$B \lesssim N \mathrm{e}^{-\frac{N\delta^p+O(1)}{2}(\beta_p^2-\beta^2(1+2\epsilon))}. \tag{2.45}$$

If p is odd and $m < 0$, $1 + f_{p,N}(m)^{-1} \leq 0$, and we immediately obtain

$$\delta_N \leq -\frac{N\delta^p\beta^2}{2}, \tag{2.46}$$

which is even better. This proves Lemma 2.5 □

Next we prove Lemma 2.6

Proof of Lemma 2.6. We have to decompose the term A further according to the value of the overlap. For α satisfying

$$\frac{1}{p} < \alpha < \frac{1}{2}, \tag{2.47}$$

we set $A = A_1 + A_2$, where

$$A_1 \equiv \mathbb{E}_{\sigma,\sigma'}\mathbb{E}\left(\mathrm{e}^{\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathbb{1}_{\{|X_\sigma-\beta\sqrt{N}|\leq\epsilon\beta\sqrt{N}\}}\mathbb{1}_{\{|X_{\sigma'}-\beta\sqrt{N}|\leq\epsilon\beta\sqrt{N}\}}\mathrm{e}^{-2NJ_N(\beta)}\mathbb{1}_{N^{-\alpha}\leq|R_N(\sigma,\sigma')|<\delta}\right), \tag{2.48}$$

and

$$A_2 \equiv \mathbb{E}_{\sigma,\sigma'}\left(\mathbb{E}\left(\mathrm{e}^{\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathbb{1}_{\{|X_\sigma-\beta\sqrt{N}|\leq\epsilon\beta\sqrt{N}\}}\mathbb{1}_{\{|X_{\sigma'}-\beta\sqrt{N}|\leq\epsilon\beta\sqrt{N}\}}\mathrm{e}^{-2NJ_N(\beta)}\right)\mathbb{1}_{|R_N(\sigma,\sigma')|<N^{-\alpha}}\right). \tag{2.49}$$

The point is that A_1 is very small, even if we drop the constraints on X_σ and $X_{\sigma'}$, whereas A_2 has to be computed precisely.

Thus, we bound A_1 by

$$0 \leq A_1 \leq \mathbb{E}_{\sigma,\sigma'}\mathbb{E}\left(\mathrm{e}^{\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathrm{e}^{-2NJ_N(\beta)}\mathbb{1}_{\{N^{-\alpha}\leq|R_N(\sigma,\sigma')|<\delta\}}\right). \tag{2.50}$$

Using the independence of the Gaussian variables

$$\mathbb{E}\left(\mathrm{e}^{\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathrm{e}^{-2NJ_N(\beta)}\right) = \prod_{K\in I_N}\mathbb{E}\left(\mathrm{e}^{\beta a_N J_K\left(\sigma_K+\sigma'_K\right)-\beta^2 a_N^2 J_K^2}\right). \tag{2.51}$$

Computing the Gaussian integrals,

$$\mathbb{E}\left(\mathrm{e}^{\beta a_N J_K\left(\sigma_K+\sigma'_K\right)-\beta^2 a_N^2 J_K^2}\right) = \mathrm{e}^{\left(1+\sigma_K\sigma'_K\right)\left(\frac{\beta^2 a_N^2}{2\beta^2 a_N^2+1}\right)-\frac{\ln(1+2\beta^2 a_N^2)}{2}}, \tag{2.52}$$

and so

$$\mathbb{E}\left(e^{\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}e^{-2NJ_N(\beta)}\right)=e^{\left(\sum_{K\in I_N}\sigma_K\sigma'_K\right)\left(\frac{\beta^2a_N^2}{2\beta^2a_N^2+1}\right)}e^{\binom{N}{p}\left(\frac{\beta^2a_N^2}{2\beta^2a_N^2+1}-\frac{\ln(1+2\beta^2a_N^2)}{2}\right)}. \tag{2.53}$$

As in (2.22), we have

$$\exp\left(\binom{N}{p}\left(\frac{\beta^2a_N^2}{2\beta^2a_N^2+1}-\frac{\ln(1+2\beta^2a_N^2)}{2}\right)\right)=\exp\left(-\beta^4Na_N^2+O\left(N^{3-2p}\right)\right). \tag{2.54}$$

Thus

$$A_1\leq\sum_{\substack{m\in\Gamma_N\\ N^{-\alpha}\leq|m|\leq\delta}}\exp\left(\frac{\beta^2Nf_{p,N}(m)}{2\beta^2a_N^2+1}\right)p_N(m)\leq\sum_{\substack{m\in\Gamma_N\\ N^{-\alpha}\leq|m|\leq\delta}}\exp\left(N\left(\frac{\beta^2f_{p,N}(m)}{2\beta^2a_N^2+1}-\frac{m^2}{2}\right)\right), \tag{2.55}$$

where the last inequality uses (2.41). Using the asymptotics for $f_{p,N}$, we get that

$$A_1\leq\sum_{\substack{m\in\Gamma_N\\ N^{-\alpha}\leq|m|\leq\delta}}\exp\left(N\frac{m^2}{2}\left(\frac{2\beta^2m^{p-2}\left(1+o_N(1)\right)}{2\beta^2a_N^2+1}-1\right)\right). \tag{2.56}$$

In the range of summation,

$$\left|\frac{2\beta^2m^{p-2}\left(1+o_N(1)\right)}{2\beta^2a_N^2+1}\right|\lesssim 2\beta^2\delta^{p-2}<1, \tag{2.57}$$

by assumption on δ, and thus, using also the lower bound on $|m|$,

$$A_1\lesssim N\exp\left(-cN^{1-2\alpha}/2\right), \tag{2.58}$$

where $c>0$.

For A_2, the constraints on the $X_\sigma,X_{\sigma'}$ can also be dropped, but this is more subtle. We write $A_2=A_{21}+R_2$, where

$$A_{21}\equiv\mathbb{E}_{\sigma,\sigma'}\left(\mathbb{E}\left(e^{\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}e^{-2NJ_N(\beta)}\right)\mathbb{1}_{|R_N(\sigma,\sigma')|<N^{-\alpha}}\right). \tag{2.59}$$

We first compute A_{21}. We set $\Gamma_N^\alpha\equiv\{m\in\Gamma_N,|m|\leq N^{-\alpha}\}$. Using (2.28), we have

$$A_{21}\exp\left(+\beta^4Na_N^2+O\left(N^{3-2p}\right)\right)\equiv\widetilde{A}_{21}=\sum_{m\in\Gamma_N^\alpha}\exp\left(\frac{\beta^2Nf_{p,N}(m)}{2\beta^2a_N^2+1}\right)p_N(m). \tag{2.60}$$

To deal with this term, we use the following standard bound for the exponential,

$$\left|\exp(\xi)-1-\xi-\frac{1}{2}\xi^2-\frac{1}{3!}\xi^3\right|\leq\frac{1}{4!}\xi^4\exp|\xi|, \tag{2.61}$$

with $\xi=\frac{\beta^2Nf_{p,N}(m)}{2\beta^2a_N^2+1}$. Notice that on Γ_N^α, $Nf_{p,N}(m)\leq N^{1-p\alpha}$, which tends to zero, as $N\uparrow\infty$. Hence, on the domain of summation of (2.60), $\exp(|\xi|)\leq e^c$. This allows us to bound A_{21} as

$$\left|\widetilde{A}_{21}-\sum_{m\in\Gamma_N^\alpha}\left(1+\xi+\frac{1}{2}\xi^2+\frac{1}{3!}\xi^3\right)p_N(m)\right|\leq\frac{1}{4!}\sum_{m\in\Gamma_N^\alpha}\xi^4e^cp_N(m). \tag{2.62}$$

Moreover, the sum over the terms on the left-hand side can be extended to sums over all of Γ_N with just an exponentially small error.

$$\left|\widetilde{A}_{21}-\sum_{m\in\Gamma_N}\left(1+\xi+\frac{1}{2}\xi^2+\frac{1}{3!}\xi^3\right)p_N(m)\right|\leq\frac{1}{4!}\sum_{m\in\Gamma_N}\xi^4e^cp_N(m)+O\left(e^{-N^{1-2\alpha}}\right). \tag{2.63}$$

The sums over the ξ^k can be computed fairly well by re-expressing them in terms of expectations over the σ. Namely

$$\sum_{m\in\Gamma_N} f_{p,N}(m)p_N(m) = \binom{N}{p}^{-1} \mathbb{E}_{\sigma,\sigma'}\left(\sum_{A\in I_N}\sigma_A\sigma'_A\right) = 0, \tag{2.64}$$

$$\sum_{m\in\Gamma_N} f_{p,N}(m)^2 p_N(m) = \binom{N}{p}^{-2} \mathbb{E}_{\sigma,\sigma'}\left(\sum_{A\in I_N}\sigma_A\sigma'_A\right)^2 = \binom{N}{p}^{-1}, \tag{2.65}$$

and, for $k \geq 3$

$$\sum_{m\in\Gamma_N} f_{p,N}(m)^k p_N(m) = \binom{N}{p}^{-k} \mathbb{E}_{\sigma,\sigma'}\left(\sum_{A\in I_N}\sigma_A\sigma'_A\right)^k \leq \binom{N}{p}^{-k} N^{pk/2}, \tag{2.66}$$

since all indices must occur alt least twice. From this we obtain

$$\widetilde{A}_{21} = 1 + \frac{\beta^4 N a_N^2}{2(1+2\beta^2 a_N^2)^2} + O\left(N^{3(1-p/2)}\right) = 1 + \frac{\beta^4 N a_N^2}{2} + O\left(N^{3(1-p/2)}\right). \tag{2.67}$$

Finally, we bound R_2. Note that

$$|R_2| \leq 2\mathbb{E}_{\sigma,\sigma'}\left(\mathbb{E}\left(e^{\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathbb{1}_{\{|X_\sigma-\beta\sqrt{N}|>\epsilon\beta\sqrt{N}\}}e^{-2NJ_N(\beta)}\right)\mathbb{1}_{|R_N(\sigma,\sigma')|<N^{-\alpha}}\right). \tag{2.68}$$

The idea here is that under the constraint on $R_N(\sigma,\sigma')$, X_σ and $X_{\sigma'}$ are almost independent. Using Hölder's inequality as before,

$$\begin{aligned}\mathbb{E}\left(e^{\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathbb{1}_{\{|X_\sigma-\beta\sqrt{N}|>\epsilon\beta\sqrt{N}\}}e^{-2NJ_N(\beta)}\right) \quad &\leq \quad \left(\mathbb{E}\left(e^{q_1\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathbb{1}_{\{|X_\sigma-\beta\sqrt{N}|>\epsilon\beta\sqrt{N}\}}\right)\right)^{\frac{1}{q_1}}\\ &\quad \times\left(\mathbb{E}\left(e^{-2q_2NJ_N(\beta)}\right)\right)^{\frac{1}{q_2}}.\end{aligned} \tag{2.69}$$

As in (2.22), we get for the second factor

$$\left(\mathbb{E}\left(e^{-2q_2NJ_N(\beta)}\right)\right)^{\frac{1}{q_2}} \leq e^{-N\beta^2+O(N^{2-p})}. \tag{2.70}$$

To deal with with first factor, we notice that $X_{\sigma'}$ can be written as

$$X_{\sigma'} = \gamma X_\sigma + \sqrt{1-\gamma^2}\xi, \tag{2.71}$$

where ξ is a normal random variable independent of X_σ and $\gamma = f_N(R_N(\sigma,\sigma'))$. Hence

$$\mathbb{E}\left(e^{q_1\beta\sqrt{N}\left(X_\sigma+X_{\sigma'}\right)}\mathbb{1}_{\{|X_\sigma-\beta\sqrt{N}|>\epsilon\beta\sqrt{N}\}}\right) = \mathbb{E}\left(e^{q_1\beta\sqrt{N}X_\sigma(1+\gamma)}\mathbb{1}_{\{|X_\sigma-\beta\sqrt{N}|>\epsilon\beta\sqrt{N}\}}\right)\mathbb{E}\left(e^{q_1\beta\sqrt{N}\sqrt{1-\gamma^2}\xi}\right). \tag{2.72}$$

Using again **Fact 1** and since $|R_N(\sigma,\sigma')| \leq N^{-\alpha}$, and that these bounds hold for all $q_1 > 1$, it follows that

$$|R_2| \leq e^{-\beta^2N\epsilon^2/2+o(N)}. \tag{2.73}$$

With these bounds on A_1 and A_2, and the bound (2.54),

$$\begin{aligned} A \quad &= \quad \left(1 + \frac{\beta^4 N a_N^2}{2(1+2\beta^2 a_N^2)^2} + O\left(N^{3(1-p/2)}\right)\right)\exp\left(-\beta^4 N a_N^2 + O\left(N^{3-2p}\right)\right)\\ &= \quad 1 - \frac{\beta^4 N a_N^2}{2} + O\left(N^{3(1-p/2)}\right). \end{aligned} \tag{2.74}$$

This implies (2.31) and concludes the proof of Lemma 2.6. □

We now conclude the proof of Proposition 2.1. Combining (2.30) and (2.31) yields

$$\mathbb{E}\left[(Z_\epsilon^{\leq})^2\right] = 1 - \frac{\beta^4 N a_N^2}{2} + O\left(N^{3-3p/2}\right). \tag{2.75}$$

Furthermore, using (2.24) we have that

$$\left(\mathbb{E}(Z_\epsilon^{\leq})\right)^2 = \left(1 - \frac{\beta^4}{4} N a_N^2 + O\left(N^{4-2p}\right)\right)^2 = 1 - \frac{\beta^4 N a_N^2}{2} + O\left(N^{4-2p}\right), \tag{2.76}$$

hence combining (2.75) and (2.76) leads to

$$\mathbb{E}\left(\Xi_\epsilon^2\right) = \frac{\mathbb{E}(Z_\epsilon^{\leq 2}) - \mathbb{E}(Z_\epsilon^{\leq})^2}{\mathbb{E}(Z_\epsilon^{\leq})^2} = \frac{O\left(N^{3-3p/2}\right)}{\mathbb{E}(Z_\epsilon^{\leq})^2}. \tag{2.77}$$

Inserting this into (2.27), we get

$$\mathbb{P}\left(\left|N^{\frac{p-2}{2}} \ln\left(1 + \Xi_\epsilon\right)\right| > \varepsilon\right) \leq 8\varepsilon^{-2} O\left(N^{1-p/2}\right), \tag{2.78}$$

which proves Lemma 2.4. □

This also concludes the proof of part (ii) of Proposition 2.1.

2.3. **Exponential concentration: proof of (i) of Proposition 2.1.**

Since

$$N^q \ln\left(\frac{\mathcal{Z}_N(\beta)}{Z_\epsilon^{\leq}}\right) = N^q \ln\left(\frac{\mathcal{Z}_N(\beta)}{\mathcal{Z}_N(\beta) - Z_\epsilon^{>}}\right) = -N^q \ln\left(1 - \frac{Z_\epsilon^{>}}{\mathcal{Z}_N(\beta)}\right), \tag{2.79}$$

the assertion 2.6 in Lemma 2.1 follows from the following lemma.

Lemma 2.7. *Assume that $\beta < \beta_p$. Then, For all $\varepsilon > 0$ there exists $\mathfrak{c} > 0$ such that*

$$\mathbb{P}\left(\frac{Z_\epsilon^{>}}{\mathcal{Z}_N(\beta)} \geq \varepsilon\right) \leq \exp(-\mathfrak{c}N). \tag{2.80}$$

Proof.

$$\begin{aligned} \mathbb{P}\left(\frac{Z_\epsilon^{>}}{\mathcal{Z}_N(\beta)} \geq \varepsilon\right) &= \mathbb{P}\left(\frac{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)} \mathbb{1}_{|-H_N(\sigma)-\beta N| > \epsilon\beta N\}}\right)}{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)} \geq \varepsilon\right) \\ &\leq \frac{1}{\varepsilon}\mathbb{E}\left(\frac{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)} \mathbb{1}_{|-H_N(\sigma)-\beta N| > \epsilon\beta N\}}\right)}{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)}\right). \end{aligned} \tag{2.81}$$

By Gaussian concentration of measure, it follows that

$$\mathbb{P}\left(\left|\ln \mathbb{E}_\sigma \mathrm{e}^{-\beta H_N(\sigma)} - \mathbb{E}\left(\ln \mathbb{E}_\sigma \mathrm{e}^{-\beta H_N(\sigma)}\right)\right| > N\beta^2 \frac{\epsilon^2}{4}\right) \leq \exp\left(-N\beta^2 \frac{\epsilon^4}{32}\right). \tag{2.82}$$

(See e.g. [5, (2.56)]).

We introduce the events

$$O_{N,\beta,\epsilon} \equiv \left\{\left|\ln \mathbb{E}_\sigma \mathrm{e}^{-\beta H_N(\sigma)} - \mathbb{E}\left(\ln \mathbb{E}_\sigma \mathrm{e}^{-\beta H_N(\sigma)}\right)\right| > N\beta^2 \tfrac{\epsilon^2}{4}\right\}, \tag{2.83}$$

and split the r.h.s. of (2.81) as

$$
\begin{aligned}
&\mathbb{E}\left(\frac{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\mathbb{1}_{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\right)}{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)}\right) &(2.84)\\
&\leq \mathbb{E}\left(\mathbb{1}_{O^c_{N,\beta,\epsilon}}\frac{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\mathbb{1}_{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\right)}{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)}\right)+\mathbb{P}(O_{N,\beta,\epsilon})\\
&\leq \mathbb{E}\left(\mathbb{1}_{O^c_{N,\beta,\epsilon}}\frac{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\mathbb{1}_{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\right)}{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)}\right)+\exp\left(-N\beta^2\frac{\epsilon^4}{32}\right),
\end{aligned}
$$

where for the first inequality we use that the quotient of the $\mathbb{E}_\sigma$-terms is smaller than one, and (2.82) is used in the last step. On the event $O^c_{N,\beta,\epsilon}$, we have that

$$
\begin{aligned}
\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right) &= \exp\left(\ln\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)-\mathbb{E}\left(\ln\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)\right)+\mathbb{E}\left(\ln\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)\right)\right)\\
&\geq \exp\left(\mathbb{E}\left(\ln\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)\right)-N\beta^2\epsilon^2/4\right). \qquad (2.85)
\end{aligned}
$$

Using this inequality

$$
\mathbb{E}\left(\mathbb{1}_{\{\{O^N_{\beta,\epsilon}\}^C\}}\frac{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\mathbb{1}_{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\right)}{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)}\right)\leq \mathrm{e}^{N\beta^2\frac{\epsilon^2}{4}}\frac{\mathbb{E}\left(\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)-N\frac{\beta^2}{2}}\mathbb{1}_{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\right)\right)}{\exp\left(\mathbb{E}\ln\mathbb{E}_\sigma\mathrm{e}^{-\beta H_N(\sigma)-N\frac{\beta^2}{2}}\right)}. \tag{2.86}
$$

By classical Gaussian estimates (**Fact I** in Appendix), the numerator on the r.h.s. above reads

$$
\mathbb{E}\left(\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)-N\frac{\beta^2}{2}}\mathbb{1}_{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\right)\right)\leq\exp\left(-N\beta^2\frac{\epsilon^2}{2}\right). \tag{2.87}
$$

Combining (2.84), (2.86) and (2.87), we obtain

$$
\mathbb{E}\left(\frac{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\mathbb{1}_{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\right)}{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)}\right)\leq\frac{\exp\left(N\beta^2\frac{\epsilon^2}{4}\right)\exp\left(-N\beta^2\frac{\epsilon^2}{2}\right)}{\exp\left(\mathbb{E}\ln\mathbb{E}_\sigma\mathrm{e}^{-\beta H_N(\sigma)-N\frac{\beta^2}{2}}\right)}+\exp\left(-N\beta^2\frac{\epsilon^4}{32}\right). \tag{2.88}
$$

It remains to bound the denominator. Note that

$$
\mathbb{E}\ln\mathbb{E}_\sigma\mathrm{e}^{-\beta H_N(\sigma)-N\frac{\beta^2}{2}}=\mathbb{E}\ln\mathbb{E}_\sigma\mathrm{e}^{-\beta H_N(\sigma)}-\ln\mathbb{E}\mathbb{E}_\sigma\mathrm{e}^{-\beta H_N(\sigma)}, \tag{2.89}
$$

so this is just the difference between the quenched and annealed free energy. In the course of the proof that these are asymptotically equal for $\beta<\beta_p$, it it actually shown that for any $\beta<\beta_p$, there exists $K>0$ such that

$$
-K\sqrt{N}<\mathbb{E}\ln\mathbb{E}_\sigma\mathrm{e}^{-\beta H_N(\sigma)}-\ln\mathbb{E}\mathbb{E}_\sigma\mathrm{e}^{-\beta H_N(\sigma)}\leq 0. \tag{2.90}
$$

(see e.g. Section 11.2 in [4]). Inserting this estimate into (2.88), it follows that

$$
\mathbb{E}\left(\frac{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\mathbb{1}_{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\right)}{\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\right)}\right)\leq\exp\left(-N\beta^2\frac{\epsilon^2}{4}+K\sqrt{N}\right)+\exp\left(-N\beta^2\frac{\epsilon^4}{32}\right). \tag{2.91}
$$

This together with the Markov inequality implies (2.80) and ends the proof of the lemma. □

Thus the proof of Lemma 2.1 is complete and this also concludes the proof of Theorem 1.1.

3. **Proof of Theorem 1.2.**

The quantity we need to control can be expressed as

$$F_N(\beta) - J_N(\beta) = \frac{1}{N} \ln (\mathcal{Z}_N(\beta)) . \tag{3.1}$$

The proof of Theorem 1.2 relies essentially on a Taylor expansion of the exponential function in $\mathcal{Z}_N(\beta)$. Recalling the definition of $J_N(\beta)$, see (1.16),

$$\mathcal{Z}_N(\beta) = \mathbb{E}_\sigma \left(\mathrm{e}^{-\beta H_N(\sigma) - \mathbb{E}_\sigma\left(\beta^2 H_N(\sigma)^2\right)/2} \right). \tag{3.2}$$

Expanding the exponential and ordering terms in powers of β, we see that

$$\mathcal{Z}_N(\beta) = T_N(\beta) + O_N(\beta^5), \tag{3.3}$$

where

$$T_N(\beta) \equiv 1 - \beta^4 \frac{\left(\mathbb{E}_\sigma H_N(\sigma)^2\right)^2}{8} - \beta^3 \frac{\mathbb{E}_\sigma \left(H_N(\sigma)^3\right)}{3!} + \beta^4 \frac{\mathbb{E}_\sigma \left(H_N(\sigma)^4\right)}{4!} . \tag{3.4}$$

Writing

$$\alpha_N(p) \ln \mathcal{Z}_N(\beta) = A_N(p) \ln (1 + \mathcal{Z}_N(\beta) - 1) , \tag{3.5}$$

with $\alpha_N(p) = A_N(p)/N$ we see that the assertion of the theorem is equivalent to

$$\alpha_N(p) (\mathcal{Z}_N(\beta) - 1) \stackrel{\mathcal{D}}{\to} \mathcal{N} \left(\mu(\beta, p), \sigma(\beta, p)^2\right). \tag{3.6}$$

The proof of Theorem 1.2 will therefore follow from the following two lemmata.

Proposition 3.1. *With the notation above, for $p > 2$ for* any $\beta > 0$,

$$\alpha_N(p) (T_N(\beta) - 1) \stackrel{\mathcal{D}}{\to} \mathcal{N} \left(\mu(\beta, p), \sigma(\beta, p)^2\right), \tag{3.7}$$

as $N \uparrow \infty$.

Proposition 3.2. *For $p > 2$ and for all $\beta < \beta_p$,*

$$\lim_{N\uparrow\infty} \alpha_N(p) \left|\mathcal{Z}_N(\beta) - T_N(\beta)\right| = 0, \text{ in probability}. \tag{3.8}$$

Remark. In view of the fact that by Lemma 2.3 $\mathcal{Z}_N(\beta)$ and $Z_\epsilon^{\leq}$ differ only by an exponentially small quantity, Proposition 3.2 is immediate if we show that

$$\lim_{N\uparrow\infty} \alpha_N(p) \left|Z_\epsilon^{\leq} - T_N(\beta)\right| = 0, \text{ in probability}. \tag{3.9}$$

The proof of these two claims is given in the next subsections. Before that, we emphasise that the different limiting pictures depending on the parity of $p > 2$ stem, in fact, from the T_N-term:

- p odd. In this case $\mathbb{E}_\sigma \left(H_N(\sigma)^3\right) = 0$ by antisymmetry (see (3.19) below), in which case
$$T_N(\beta) = 1 - \beta^4 \frac{\mathbb{E}_\sigma \left(H_N(\sigma)^2\right)^2}{8} + \beta^4 \frac{\mathbb{E}_\sigma \left(H_N(\sigma)^4\right)}{4!}. \tag{3.10}$$
This should be contrasted to
- p even. We will see in the course of the proof that the only relevant term is, as a matter of fact, the third moment, with the second and fourth moments contributing nothing due to a "wrong" blow-up. In other words, it will become clear that
$$T_N(\beta) = 1 + \beta^3 \frac{\mathbb{E}_\sigma \left(-H_N(\sigma)^3\right)}{3!} + \text{"vanishing corrections"}. \tag{3.11}$$

We prove Propositions 3.1 and 3.2 in the remainder of this paper. As a first step, in Section 3.1 below we provide some explicit formulas for the moments of $\mathbb{E}_\sigma H^k, k = 2, 3, 4$ which appear in the definition of $T_N(\beta)$. Proposition 3.1 for odd p is then proven in Section 3.2 below, whereas the case of p even in Section 3.3; the proof of Proposition 3.2 for even p is given in Section 3.4 and the proof for the odd p case is finally given in Section 3.5.

3.1. **Explicit representations of quenched moments.** In the sequel we use the following abbreviation when summing over multi-indices $A, B \in I_N$.

$$\sum_{(\neq)} J_A J_B \mathbb{E}_\sigma(\sigma_A\sigma_B) \equiv \sum_{A,B\in I_N : A\neq B} J_A J_B \mathbb{E}_\sigma(\sigma_A\sigma_B), \tag{3.12}$$

and similarly for sums involving a higher number of multi-indices, in which case we mean that all multi-indices involved must be different.

For the different terms appearing in $T_N(\beta)$, taking into account cancellations due to the averages over σ, we have the following representations.

Lemma 3.3. *We have*

$$\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right) = a_N^3 \sum_{A,B,C\in I_N} J_A J_B J_C \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\right) = a_N^3 \sum_{(\neq)} J_A J_B J_C \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\right). \tag{3.13}$$

and

$$-\frac{1}{8}\mathbb{E}_\sigma\left(H_N(\sigma)^2\right)^2 + \frac{1}{4!}\mathbb{E}_\sigma\left(H_N(\sigma)^4\right) = -\frac{a_N^4}{12}\sum_{A\in I} J_A^4 + \mathcal{H}_4, \tag{3.14}$$

where

$$\mathcal{H}_4 \equiv \frac{a_N^4}{4!}\sum_{(\neq)} J_A J_B J_C J_D \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right). \tag{3.15}$$

Proof. Eq. (3.13) is straightforward. An elementary computations shows that

$$-\mathbb{E}_\sigma\left(H_N(\sigma)^2\right)^2 = -a_N^4 \sum_{A,B\in I_N} J_A^2 J_B^2 = -a_N^4 \sum_{(\neq)} J_A^2 J_B^2 - a_N^4 \sum_{A\in I_N} J_A^4. \tag{3.16}$$

The fourth moment gives

$$\mathbb{E}_\sigma\left(H_N(\sigma)^4\right) = a_N^4 \sum_{A,B,C,D\in I_N} J_A J_B J_C J_D \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right). \tag{3.17}$$

We now rearrange the summation according to the possible sub-cases: *i)* four multi-indices come in two distinct pairs (say $A = B$ and $C = D$ but $A \neq C$): in this case $\mathbb{E}_\sigma\sigma_A\sigma_B\sigma_C\sigma_D = \mathbb{E}_\sigma\sigma_A^2\sigma_C^2 = 1$; *ii)* all four multi-indices coincide, in which case $\mathbb{E}_\sigma\sigma_A\sigma_B\sigma_C\sigma_D = \mathbb{E}_\sigma\sigma_A^4 = 1$; *iii)* at least one multi-index is different from all the others. In this case the only non-vanishing contribution comes if four multi-indices are different. Hence

$$\mathbb{E}_\sigma\left(H_N(\sigma)^4\right) = 3a_N^4 \sum_{(\neq)} J_A^2 J_C^2 + a_N^4 \sum_{A\in I_N} J_A^4 + a_N^4 \sum_{(\neq)} J_A J_B J_C J_D \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right), \tag{3.18}$$

where for the first term on the right we use that there are $\binom{4}{2} = 3$ ways to choose the pairs. Combining (3.16) and (3.18) yields the claim of the lemma. □

3.2. **Proof of Proposition 3.1:** p **odd.** We first observe that

$$\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right) = a_N^3 \sum_{A,B,C\in I_N} J_A J_B J_C \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\right) = 0, \tag{3.19}$$

since $\sigma_A\sigma_B\sigma_C$ is a product of an *odd* number of spins, and hence its expectation vanishes. Combining Lemma 3.3 and (3.19), it follows that

$$\alpha_N(p)\left(T_N(\beta)-1\right) = N^{p-2}\left(-\frac{\beta^4 a_N^4}{12}\sum_{A\in I} J_A^4 + \beta^4\mathcal{H}_4\right). \tag{3.20}$$

First note that

$$N^{p-2}\left(-\frac{\beta^4 a_N^4}{12}\sum_{A\in I} J_A^4\right) = -N^{p-2}\frac{\beta^4 N^2}{12\binom{N}{p}}\frac{1}{\binom{N}{p}}\sum_{A\in I} J_A^4 \to -\frac{\beta^4 p!}{4}, \text{a.s.}, \tag{3.21}$$

as $N\uparrow\infty$ by the strong law of large numbers. It remains to prove that $N^{p-2}\mathcal{H}_4$ converges to a Gaussian with mean zero and variance $\sigma(\beta,p)^2$. This will be done by proving that the moments of $N^{p-2}\mathcal{H}_4$ converge to those of the Gaussian. We break this up into a series of lemmata.

Lemma 3.4. *(Second moment / variance). For any $\beta \geq 0$ and any $p \geq 3$,*

$$\lim_{N\to+\infty}\beta^8\mathbb{E}\left(\left(N^{p-2}\mathcal{H}_4\right)^2\right) = \sigma(\beta,p)^2, \tag{3.22}$$

Lemma 3.5. *(Even moments). For any $\beta \geq 0$, and p odd, and for all $k\in\mathbb{N}$,*

$$\lim_{N\to+\infty}\beta^{8k}E\left(\left(N^{p-2}\mathcal{H}_4\right)^{2k}\right) = \frac{(2k)!}{2^k k!}\sigma(\beta,p)^{2k}. \tag{3.23}$$

Lemma 3.6. *(Vanishing of odd moments). For any $\beta \geq 0$, p odd and for all $k\in\mathbb{N}$,*

$$\lim_{N\to+\infty}\mathbb{E}\left(\left(N^{p-2}\mathcal{H}_4\right)^{2k+1}\right) = 0. \tag{3.24}$$

The remainder of this subsection is devoted to the proofs of these lemmata, which combined imply Proposition 3.1 for p odd.

Proof of Lemma 3.4. We have that

$$\begin{aligned}
\mathbb{E}\left(\mathcal{H}_4^2\right) &= \frac{a_N^8}{4!^2}\sum_{\substack{A,B,C,D\in I_N\\(\neq)}}\sum_{\substack{E,F,G,H\in I_N\\(\neq)}}\mathbb{E}\left(J_AJ_B\dots J_H\right)\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\mathbb{E}_{\sigma'}\left(\sigma'_E\sigma'_F\sigma'_G\sigma'_H\right)\\
&= 4!\frac{a_N^8}{4!^2}\sum_{(\neq)}\mathbb{E}\left(J_A^2J_B^2J_C^2J_D^2\right)\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\mathbb{E}_{\sigma'}\left(\sigma'_A\sigma'_B\sigma'_C\sigma'_D\right)\\
&= \frac{a_N^8}{4!}\sum_{(\neq)}\mathbb{E}_{\sigma,\sigma'}\left(\sigma_A\sigma_B\sigma_C\sigma_D\sigma'_A\sigma'_B\sigma'_C\sigma'_D\right).
\end{aligned} \tag{3.25}$$

Here we used that in order to get a non-vanishing contributions, all the multi-indices in the first sum must be paired with one in the second sum. The number of such pairings is 4!.

Next we express $\mathbb{E}\left(\mathcal{H}_4^2\right)$ as a function of the overlaps.

$$\begin{aligned}
\mathbb{E}\left(\mathcal{H}_4^2\right) &= \frac{a_N^8}{4!}\Big[\sum_{A,B,C,D\in I_N}\mathbb{E}_{\sigma,\sigma'}\left(\sigma_A\sigma_B\sigma_C\sigma_D\sigma'_A\sigma'_B\sigma'_C\sigma'_D\right) - 3\sum_{\substack{A,B\in I_N\\(\neq)}}\mathbb{E}_{\sigma,\sigma'}\left(\sigma_A^2\sigma_B^2\sigma_A'^2\sigma_B'^2\right)\\
&\quad -\sum_{A\in I_N}\mathbb{E}_{\sigma,\sigma'}\left(\sigma_A^4\sigma_A'^4\right)\Big] \qquad (3.26)\\
&= \frac{a_N^8}{4!}\left[\sum_{A,B,C,D\in I_N}\mathbb{E}_{\sigma,\sigma'}\left(\sigma_A\sigma_B\sigma_C\sigma_D\sigma'_A\sigma'_B\sigma'_C\sigma'_D\right) - 3\left(\binom{N}{p}^2-\binom{N}{p}\right)-\binom{N}{p}\right],
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathbb{E}\left(\mathcal{H}_4^2\right) &= \frac{a_N^8}{4!}\mathbb{E}_{\sigma,\sigma'}\left(\left(\sum_{A\in I_N}\sigma_A\sigma'_A\right)^4\right) - \frac{3a_N^8}{4!}\binom{N}{p}^2 + \frac{2\beta^8a_N^8}{4!}\binom{N}{p}\\
&= \frac{1}{4!}\sum_{m\in\Gamma_N}\left(Nf_N^p(m)\right)^4 p_N(m) - \frac{N^4}{8\binom{N}{p}^2} + O(N^{4-3p}), \qquad (3.27)
\end{aligned}$$

where we used (2.28). Collecting the leading terms, we see that

$$\mathbb{E}\left(\left(N^{p-2}\mathcal{H}_4\right)^2\right) = \frac{1}{4!}\sum_{m\in\Gamma_N}\left(N^{\frac{p}{2}}f_N^p(m)\right)^4 p_N(m) - \frac{p!^2}{8} + o(1). \qquad (3.28)$$

Furthermore, by (1.4), we have that

$$N^{\frac{p}{2}}f_N^p(m) = \sum_{k=0}^{[p/2]} d_{p-2k}\left(\sqrt{N}m\right)^{p-2k}(1+O(1/N)), \qquad (3.29)$$

and using this in the sum on the r.h.s. of (3.28) yields

$$\mathbb{E}\left(\left(N^{p-2}\mathcal{H}_4\right)^2\right) = \frac{1}{4!}\sum_{m\in\Gamma_N}\left(\sum_{k=0}^{[p/2]} d_{p-2k}\left(\sqrt{N}m\right)^{p-2k}\right)^4 p_N(m)\left(1+O\left(\frac{1}{N}\right)\right) - \frac{p!^2}{8} + o_N(1)\,. \qquad (3.30)$$

By Taylor-expanding in $m=0$, it can be checked that

$$p_N(m) = \frac{2}{\sqrt{2\pi N}}\mathrm{e}^{-Nm^2/2}[1+o_N(1)]. \qquad (3.31)$$

It follows that the sum in (3.30) converges to an integral, namely,

$$\begin{aligned}
\lim_{N\uparrow\infty}\mathbb{E}\left(\left(N^{p-2}\mathcal{H}_4\right)^2\right) &= \frac{1}{12\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\sum_{k=0}^{[p/2]} d_{p-2k}m^{p-2k}\right)^4 \mathrm{e}^{-\frac{m^2}{2}}dm - \frac{p!^2}{8}\\
&= \beta^{-8}\sigma(\beta,p)^2. \qquad (3.32)
\end{aligned}$$

This proves the lemma. ◻

Proof of Lemma 3.5. The $2k$-th moments of $\mathcal{H}_4$ can be written as

$$\mathbb{E}\left(\mathcal{H}_4^{2k}\right) = \frac{a_N^{8k}}{4!^{2k}}\mathbb{E}\left(\left(\sum_{(\neq)}J_AJ_BJ_CJ_D\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\right)^{2k}\right). \qquad (3.33)$$

and

$$\mathbb{E}\left(\left(\sum_{\substack{A,B,C,D\in I_N\\(\neq)}} J_A J_B J_C J_D \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\right)^{2k}\right) =$$

$$\prod_{i=1}^{2k}\sum_{\substack{A_i,B_i,C_i,D_i\in I_N\\(\neq)}}\mathbb{E}\left(\prod_{i=1}^{2k} J_{A_i}J_{B_i}J_{C_i}J_{D_i}\right)\prod_{i=1}^{2k}\mathbb{E}_\sigma\left(\sigma_{A_i}\sigma_{B_i}\sigma_{C_i}\sigma_{D_i}\right). \tag{3.34}$$

Since the averages of odd powers of the random variables J vanish, only terms in the sums over the multi-indices in (3.34) give a non-zero contribution where each multi-index occurs at least twice. Moreover, the leading order contribution comes from terms where each multi-index occurs *exactly* twice and where these pairings take place between the multi-indices of two indices i and j. We say a *pairing between the sums i and j takes place* as soon as $(A_i, B_i, C_i, D_i) = (\pi[A_j], \pi[B_j], \pi[C_j], \pi[D_j])$ where π is any permutation [1] on (A_j, B_j, C_j, D_j). Since there are $\frac{(2k)!}{k!2^k}$ different ways to construct such sum-pairings, we re-write the right-hand side of (3.34) as

$$\frac{4!^k(2k)!}{k!2^k}\sum_{(\neq)}\prod_{i=1}^{k}\mathbb{E}\left(J_{A_i}^2 J_{B_i}^2 J_{C_i}^2 J_{D_i}^2\right)\left(\mathbb{E}_\sigma\left(\sigma_{A_i}\sigma_{B_i}\sigma_{C_i}\sigma_{D_i}\right)\right)^2 + R_N(2k) \equiv P_N(2k) + R_N(2k).$$

The first term can be written as

$$P_N(2k) = \frac{4!^k(2k)!}{k!2^k}\sum_{(\neq)}\prod_{i=1}^{k}\left(\mathbb{E}_\sigma\left(\sigma_{A_i}\sigma_{B_i}\sigma_{C_i}\sigma_{D_i}\right)\right)^2. \tag{3.35}$$

This term will converge to the appropriate moment of the Gaussian, whereas the R_N-term tend to zero.

Lemma 3.7. *With the notation above,*

$$\lim_{N\uparrow\infty}\frac{N^{(2pk-4k)}a_N^{8k}\beta^{8k}}{4!^{2k}}P_N(2k) = \frac{(2k)!}{k!2^k}\sigma(\beta,p)^{2k}. \tag{3.36}$$

Proof. It is elementary to see that

$$\sum_{(\neq)}\prod_{i=1}^{k}\mathbb{E}_\sigma\left(\sigma_{A_i}\sigma_{B_i}\sigma_{C_i}\sigma_{D_i}\right)^2 = \left(\sum_{(\neq)}\left(\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\right)^2\right)^k\left(1 + O(N^{-p})\right). \tag{3.37}$$

Recalling (3.25),

$$\sum_{(\neq)}\left(\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\right)^2 = \frac{4!}{a_N^8}\mathbb{E}\left(\mathcal{H}_4^2\right). \tag{3.38}$$

Putting these observations together and using (3.32), we arrive at the assertion of the lemma. □

We now turn to the remainder term.

Lemma 3.8.

$$\lim_{N\uparrow\infty}\frac{N^{(2pk-4k)}a_N^{8k}}{4!^{2k}}R_N(2k) = 0. \tag{3.39}$$

[1] note that we have 4! possible permutations.

Proof. Recall that the sums in (3.34) run over $8k$ multi-indices which by the pairing condition due to the J is reduced to $4k$ multi-indices. In $P_N(2k)$, there are indeed that many sums. We must show that in what is left, i.e. if pairings occur that involve more than two groups, the effective number os summations is further reduced. This means that there are terms where (double) products of the following type appear:

(1)
$$\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\mathbb{E}_\sigma\left(\sigma_A\sigma_E\sigma_F\sigma_G\right),$$
where (E, F, G) do not coincide with any of the multi-indices (A, B, C, D) or

(2)
$$\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_E\sigma_F\right),$$
where (E, F) do not coincide with any of the multi-indices (A, B, C, D) [2] or

(3)

sums which appear in pairs but at least one of the pairs coincide.

The last case it trivially of lower order.

We first look at the terms of type (1). They are of the form

$$\widetilde{\sum}_{(1)}\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\mathbb{E}_\sigma\left(\sigma_A\sigma_E\sigma_F\sigma_G\right)\prod_{i=1}^{2k-2}\mathbb{E}_\sigma\left(\sigma_{A_i}\sigma_{B_i}\sigma_{C_i}\sigma_{D_i}\right), \tag{3.40}$$

where the sum is over at most $4k$ different multi-indices where moreover A, B, C, D, E, F, G respect the condition stated under (1) and of course the multi-indices with same index i are all different. We first note that

$$\sum_{\substack{A,B,C,D\in I_N\\(\neq)}}\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\lesssim N^{2p}, \tag{3.41}$$

since the expectation over σ vanishes unless all σ_i appearing in the product come in pairs. Thus, we may run A over all N^p values. Then B, C, D may each match k_B, k_C and k_D with $k_B + k_C + k_D = p$ of the indices of A. Further, C may in addition match ℓ_C of the $p - k_B$ free indices of B. Then D must match the remaining $p - k_B - k_C$ unmatched indices of A, the $p - k_B - \ell_C$ unmatched indices of B and the $p - k_C - \ell_C$ free indices of C. This leaves N^{p-k_B} choices for B, $N^{p-k_C-\ell_C}$ choices for C, and just one for D. Clearly, $\ell_C = k_D$, since D must match the $p - k_B - k_C$ unmatched indices of A. Thus, the number of choices for the four multi-indices is $N^{p+p-k_B+p-k_C-\ell_C} = N^{2p}$. If in addition one of the multi-indices is fixed, we are left with

$$\sum_{\substack{B,C,D\in I_N\\(\neq)}}\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\lesssim N^{p}, \tag{3.42}$$

where the B, C, D must also be different from A. If two multi-indices are fixed,

$$\sum_{\substack{C,D\in I_N\\(\neq)}}\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\lesssim N^{p-1}. \tag{3.43}$$

This bound comes from the case when B matches the largest possible number of the indices in A, namely $N - 1$. In that case, C has to just match the one remaining index from A, leaving N^{p-1} choices that then have to be matched by D. Finally, if all four multi-indices are fixed there is only one contribution. We see that the cost of fixing one multi-index

[2] Note that $\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_E\right)$ implies that $E = D$ and is thus not a particular case.

ist at least $N^{-p/2}$ which is achieved only if four are fixed in the same pack of four (which corresponds to the terms in $P_N(2k)$).

Let us now return to the sum (3.40),

$$\widetilde{\sum}_{(1)} \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\mathbb{E}_\sigma\left(\sigma_A\sigma_E\sigma_F\sigma_G\right)\prod_{i=1}^{2k-2}\mathbb{E}_\sigma\left(\sigma_{A_i}\sigma_{B_i}\sigma_{C_i}\sigma_{D_i}\right). \tag{3.44}$$

The sum over the seven multi-indices A, B, C, D, E, F, G gives at most N^{3p} terms: The sum over A gives N^p, and then, according to the discussion above, the B, C, D and the E, F, G N^p each. The remaining sum is over $4(2k-2)$ multi-indices, of which 6 have to be matched to B, C, D, E, F, G, and all others must be paired. This leaves $4k-7$ sums over multi-indices to be summed, which gives due to the constraints created by the σ-sums at most $N^{p(2k-7/2)}$ terms. So overall, (3.44) is bounded by a constant times $N^{p(2k-1/2)} \ll N^{2kp}$.

Terms of Type (2) are of the form

$$\widetilde{\sum}_{(2)} \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_E\sigma_F\right)\prod_{i=1}^{2k-2}\mathbb{E}_\sigma\left(\sigma_{A_i}\sigma_{B_i}\sigma_{C_i}\sigma_{D_i}\right). \tag{3.45}$$

To bound the sum over the first six multi-indices, we have to be more careful. First, there are N^p choices for A. Then, if we choose B such that k_B indices match those of B, there are N^{p-k_B} choices for B. Finally, we must choose k_C and ℓ_C as in the discussion above, thus that $k_B + k_C + \ell_C = p$, and equally k_E and ℓ_E with the same property. This allows N^{p-k_B} choices for each of these multi-indices. Finally, E and F are determined. Altogether, this leaves $N^{4p-k_B-k_C-\ell_C-k_E-\ell_E} = N^{2p+k_B}$ terms, for k_B given. But since $B \neq A$, $k_B \leq p-1$, so that the sum over these 6 indices contribute at most $O(N^{3p-1})$ terms. From the remaining $4(2k-2)$ multi-indices, four are fixed to match C, D, E, F, and all others must be paired. This leaves $2(2k-3)$ free multi-indices which can at most contribute $N^{p(2k-3)}$ terms. So in all the sum in (3.45) is bounded by $Cont.N^{2kp-1}$, which is again of lower order than N^{2kp}.

Finally, if any multi-index occurs four times, we loose a factor of N^{2p} and also these terms are negligable. Combining these observations we have proven the lemma. □

The assertion of Lemma 3.5 follows immediately. □

Proof of Lemma 3.6. In the case of odd moments, pairing of the multi-indices between always just two blocks is obviously impossible, so that the terms that contributed to the leading $P_N(2k+1)$ do not exist. Thus

$$\mathbb{E}\left(\left(N^{p-2}\mathcal{H}_4\right)^{2k+1}\right) = \frac{N^{(p-2)(2k+1)}a_N^{4(2k+1)}}{4!^{2k+1}}R_N(2k+1) \lesssim \frac{1}{N^{(2k+1)p}}R_N(2k+1). \tag{3.46}$$

By the same arguments as in the proof of Lemma 3.8, $R_N(2k+1)$ is of smaller order than $N^{(2k+1)p}$ and hence the right-hand side of (3.46) tends to zero. This proves Lemma 3.6. □

This also concludes the proof of Proposition 3.1 for p odd.

3.3. **Proof of Proposition 3.1:** *p* **even.**

Recall that for p even,

$$\alpha_N(p)\left(T_N(\beta)-1\right) = \beta^4 N^{\left(\frac{3p}{4}-\frac{3}{2}\right)}\left(\frac{-\mathbb{E}_\sigma\left(H_N(\sigma)^2\right)^2}{8} + \frac{\mathbb{E}_\sigma\left(H_N(\sigma)^4\right)}{4!}\right) + \beta^3 N^{\left(\frac{3p}{4}-\frac{3}{2}\right)}\frac{\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)}{3!}. \tag{3.47}$$

We first show that only the last term is relevant.

Lemma 3.9.

$$\lim_{N\uparrow\infty} N^{\left(\frac{3p}{4}-\frac{3}{2}\right)}\left(-\frac{\mathbb{E}_\sigma\left(H_N(\sigma)^2\right)^2}{8}+\frac{\mathbb{E}_\sigma\left(H_N(\sigma)^4\right)}{4!}\right)=0. \tag{3.48}$$

Proof. By Lemma 3.3,

$$N^{\left(\frac{3p}{4}-\frac{3}{2}\right)}\left(-\frac{\mathbb{E}_\sigma\left(H_N(\sigma)^2\right)^2}{8}+\frac{\mathbb{E}_\sigma\left(H_N(\sigma)^4\right)}{4!}\right)=-N^{\left(\frac{3p}{4}-\frac{3}{2}\right)}\frac{a_N^4}{12}\sum_{A\in I}J_A^4+N^{\left(\frac{3p}{4}-\frac{3}{2}\right)}\mathcal{H}_4. \tag{3.49}$$

By the law of large numbers (see (3.21)), the first term in the right converges to zero in probability. By Lemma 3.4, $N^{p-2}\mathcal{H}_4$ converges to a constant in L^2. Since $\frac{3p}{4}-\frac{3}{2}<p-2$ if $p>2$, this implies that the last term in (3.49) also converges to zero in probability. This proves the lemma. □

Thus, it only remains to prove that

$$\beta^3 N^{\left(\frac{3p}{4}-\frac{3}{2}\right)}\frac{\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)}{3!}\xrightarrow{\mathcal{D}}\mathcal{N}(0,\sigma(\beta,p)^2). \tag{3.50}$$

to conclude the proof of Proposition 3.1. We break this up into three lemmata as in the odd case.

Lemma 3.10. *(Second moment). For any $\beta\geq 0$,*

$$\lim_{N\to+\infty}\beta^6\mathbb{E}\left(\left(N^{\left(\frac{3p}{4}-\frac{3}{2}\right)}\mathbb{E}_\sigma\left(\frac{-H_N(\sigma)^3}{3!}\right)\right)^2\right)=\sigma(\beta,p)^2. \tag{3.51}$$

Lemma 3.11. *(Even moments). For any $\beta\geq 0$,*

$$\lim_{N\to+\infty}\beta^{6k}\mathbb{E}\left(\left(N^{\left(\frac{3p}{4}-\frac{3}{2}\right)}\mathbb{E}_\sigma\left(\frac{-H_N(\sigma)^3}{3!}\right)\right)^{2k}\right)=\frac{(2k)!}{2^k k!}\sigma(\beta,p)^{2k}. \tag{3.52}$$

Lemma 3.12. *(Odd moments). For any $\beta\geq 0$,*

$$\lim_{N\to+\infty}\mathbb{E}\left(\left(N^{\left(\frac{3p}{4}-\frac{3}{2}\right)}\mathbb{E}_\sigma\left(\frac{-H_N(\sigma)^3}{3!}\right)\right)^{2k+1}\right)=0. \tag{3.53}$$

Proof of Lemma 3.10. We have that

$$\begin{aligned}&\mathbb{E}\left(\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)^2\right)\\&=a_N^6\sum_{\substack{A,B,C\in I_N(\neq)\\D,E,F\in I_N(\neq)}}\mathbb{E}\left(J_AJ_BJ_CJ_DJ_EJ_F\right)\mathbb{E}_{\sigma,\sigma'}\left(\sigma_A\sigma_B\sigma_C\sigma'_D\sigma'_E\sigma'_F\right).\end{aligned} \tag{3.54}$$

We rearrange the summation according to the possible sub-cases: *i)* all four multi-indices coincide, *ii)* four multi-indices coincide and two multi-indices come in a distinct pair; *iii)* six multi-indices come in three different pairs. Thus the right-hand side of (3.54) equals

$$\begin{aligned}&a_N^6\mathbb{E}\left(J^6\right)\sum_{A\in I_N}\mathbb{E}_{\sigma,\sigma'}\left(\sigma_A\sigma'_A\right)+a_N^6\binom{6}{2}\mathbb{E}\left(J^4\right)\mathbb{E}\left(J^2\right)\sum_{A\neq B\in I_N}\mathbb{E}_{\sigma,\sigma'}\left(\sigma_A\sigma'_B\right)\\&+6a_N^6\mathbb{E}\left(J^2\right)^3\sum_{A,B,C\in I_N(\neq)}\mathbb{E}_{\sigma,\sigma'}\left(\sigma_A\sigma_B\sigma_C\sigma'_A\sigma'_B\sigma'_C\right)\\&=6a_N^6\sum_{A,B,C\in I_N}\mathbb{E}_{\sigma,\sigma'}\left(\sigma_A\sigma_B\sigma_C\sigma'_A\sigma'_B\sigma'_C\right),\end{aligned} \tag{3.55}$$

where the factor 6 accounts for the 3! possible pairings that all give the same contribution. In the last line we dropped the condition ($\neq$), since all terms where this is not satisfied vanish. We conclude that

$$\mathbb{E}\left(\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)^2\right) = 6a_N^6 \mathbb{E}_{\sigma,\sigma'}\left[\left(\sum_{A\in I_N} \sigma_A\sigma'_A\right)^3\right] = 6\sum_{m\in\Gamma_N}\left(Nf_N^p(m)\right)^3 p_N(m). \tag{3.56}$$

From here we get

$$\mathbb{E}\left(N^{2\left(\frac{3p}{4}-\frac{3}{2}\right)}\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)^2\right) = 3!\sum_{m\in\Gamma_N}\left(N^{\frac{p}{2}}f_N^p(m)\right)^3 p_N(m). \tag{3.57}$$

Exactly as in the proof of Lemma 3.4 it now follows that

$$\lim_{N\uparrow\infty}\mathbb{E}\left(N^{2\left(\frac{3p}{4}-\frac{3}{2}\right)}\frac{\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)^2}{3!^2}\right) = \frac{1}{3\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\sum_{k=0}^{[p/2]} d_{p-2k}m^{p-2k}\right)^3 e^{-\frac{m^2}{2}}dm, \tag{3.58}$$

which proves the lemma. □

Proof of Lemma 3.11. For $k>1$, we consider

$$\mathbb{E}\left(\left(N^{\left(\frac{3p}{4}-\frac{3}{2}\right)}\mathbb{E}_\sigma\left(\frac{H_N(\sigma)^3}{3!}\right)\right)^{2k}\right) = \frac{N^{\left(\frac{3pk}{2}-3k\right)}a_N^{6k}}{3!^{2k}}\mathbb{E}\left(\left(\sum_{(\neq)} J_AJ_BJ_C\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\right)\right)^{2k}\right). \tag{3.59}$$

Expanding the $2k$-moment inside the expectation yields

$$\begin{aligned}&\mathbb{E}\left(\left(\sum_{(\neq)} J_{A_1}J_{B_1}J_{C_1}\mathbb{E}_\sigma\left(\sigma_{A_1}\sigma_{A_2}\sigma_{A_3}\right)\right)^{2k}\right) = \\ &\prod_{i=1}^{2k}\sum_{(\neq)}\mathbb{E}\left(\prod_{i=1}^{2k} J_{A_i}J_{B_i}J_{C_i}\right)\prod_{i=1}^{2k}\mathbb{E}_\sigma\left(\sigma_{A_i}\sigma_{B_i}\sigma_{C_i}\right).\end{aligned} \tag{3.60}$$

We now proceed as in the case p odd. The principal term in the sum comes form the multi-indices within two blocks i,j are $(A_i,B_i,C_i) = (\pi[A_j],\pi[B_j],\pi[C_j])$ matched. Since there are $\frac{(2k)!}{k!2^k}$ different ways to construct such sum-pairings, we re-write the right-hand side of (3.60) as

$$\begin{aligned}&\frac{3!^k(2k)!}{k!2^k}\sum_{\substack{A_1,B_1,C_1\ldots A_k,B_k,C_k\in I_N\\(\neq)}}\prod_{i=1}^{k}\mathbb{E}\left(J_{A_i}^2J_{B_i}^2J_{C_i}^2\right)\prod_{i=1}^{k}\mathbb{E}_\sigma\left(\sigma_{A_i}\sigma_{B_i}\sigma_{C_i}\right)^2 + R_N(2k)\\ &\equiv P_N(2k)+R_N(2k).\end{aligned} \tag{3.61}$$

As in the odd case, we have the following results.

Lemma 3.13. *With the notation above,*

$$\lim_{N\uparrow\infty}N^{\left(\frac{3pk}{2}-3k\right)}\frac{\beta^{6k}a_N^{6k}}{3!^{2k}}P_N(2k) = \frac{(2k)!}{k!2^k}\sigma(\beta,p)^{2k}. \tag{3.62}$$

Lemma 3.14.

$$\lim_{N\uparrow\infty}N^{\left(\frac{3pk}{2}-3k\right)}\frac{a_N^{6k}}{3!^{2k}}R_N(2k) = 0. \tag{3.63}$$

Proof of Lemma 3.13. The proof is completely analogous to that of Lemma 3.7 and will be omitted. □

Proof of Lemma 3.14. The non-trivial terms that appear in the expression for $R_N(2k)$ must contain a term of the form

$$\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\right)\mathbb{E}_\sigma\left(\sigma_A\sigma_D\sigma_E\right), \tag{3.64}$$

where (D,E) do not coincide with any of the multi-indices (A,B,C) [3] That is, we have to control sums of the form

$$\widetilde{\sum}_{(1)}\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\right)\mathbb{E}_\sigma\left(\sigma_A\sigma_D\sigma_E\right)\prod_{i=1}^{2k-2}\mathbb{E}_\sigma\left(\sigma_{A_i}\sigma_{B_i}\sigma_{C_i}\right), \tag{3.65}$$

where A,B,C,D,E are as above and all multi-indices must be paired. By a computation analogous to that in the proof of Lemma 3.7, we get that

$$\sum_{\substack{A,B,C\in I_N\\(\neq)}}\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\right)\lesssim N^{\frac{3p}{2}}. \tag{3.66}$$

Looking at (3.64), we see that the sum over the (A,B,C,D,E) produces $O(N^{2p})$ terms. Of the remaining $3(2k-2)$ multi-indices, four must match B,C,D,E while the remaining ones must be paired. This leaves $(3k-5)$ free multi-indices to sum over. This yields at most $N^{p(3k-5)/2}$ terms, so that altogether the sum in (3.65) is of order at most $N^{p(3k-1)/2}$. Inserting this into (3.63) shows that the left-hand side is of order $N^{-p/2}$ and converges to zero as claimed. □

Lemma 3.13 and Lemma 3.14 yield the assertion of Lemma 3.11. □

Proof of Lemma 3.12. $\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)^{2k+1}$ is a sum of a product of $6k+3$ standard normal random variables, which is an odd number: At least one of the J. will be to the power of an odd number. The expectation value of $\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)^{2k+1}$ with respect to $\mathbb{E}$ is thus equal to 0. □

3.4. **Proof of Proposition 3.2:** p **even.** We want to show that

$$\lim_{N\uparrow\infty}N^{3p/4-3/2}\left|\mathcal{Z}_N(\beta)-T_N(\beta)\right|=0. \tag{3.67}$$

Using the definition of $T_N(\beta)$

$$\begin{aligned}\left|\mathcal{Z}_N(\beta)-T_N(\beta)\right| &\leq \left|\mathcal{Z}_N(\beta)-1-\frac{\beta^3}{3!}\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right|+\left|\frac{\beta^4}{8}\mathbb{E}_\sigma\left(H_N(\sigma)^2\right)^2-\frac{\beta^4}{4!}\mathbb{E}_\sigma\left(H_N(\sigma)^4\right)\right|\\ &\leq \left|Z_\epsilon^{\leq}-\mathcal{Z}_N(\beta)\right|+\left|\mathbb{E}\left(Z_\epsilon^{\leq}\right)-1\right|+\left|Z_\epsilon^{\leq}-\mathbb{E}\left(Z_\epsilon^{\leq}\right)-\frac{\beta^3}{3!}\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right|\\ &+\left|\frac{\beta^4}{8}\mathbb{E}_\sigma\left(H_N(\sigma)^2\right)^2-\frac{\beta^4}{4!}\mathbb{E}_\sigma\left(H_N(\sigma)^4\right)\right|.\end{aligned} \tag{3.68}$$

The first term in the second line is negligible by Lemma (2.3), the second by (2.18) together with Lemma (2.3). By Lemma 3.9, the last term on the right of (3.68) will vanish if it is inserted into (3.67). To control the remaining third term, we bound its second moment,

$$\begin{aligned}\mathbb{E}\left(\left|Z_\epsilon^{\leq}-\mathbb{E}(Z_\epsilon^{\leq})-\frac{\beta^3}{3!}\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right|^2\right) &= \mathbb{E}\left(\left(Z_\epsilon^{\leq}\right)^2-\left(\mathbb{E}(Z_\epsilon^{\leq})\right)^2\right)\\ +\frac{\beta^6}{3!^2}\mathbb{E}\left(\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)^2\right)&-\frac{2\beta^3}{3!}\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\left(Z_\epsilon^{\leq}-\mathbb{E}\left(Z_\epsilon^{\leq}\right)\right)\right).\end{aligned} \tag{3.69}$$

[3]Note that $\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\right)\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_D\right)$ implies that $C=D$ and is thus not a particular case.

$\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right) = 0$ by symmetry. Therefore, the right-hand side of (3.69) is equal to

$$\mathbb{E}\left((Z_\epsilon^{\leq})^2\right)-\left(\mathbb{E}\left(Z_\epsilon^{\leq}\right)\right)^2-\frac{\beta^6}{3!^2}\mathbb{E}\left(\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)^2\right)-\frac{2\beta^3}{3!}\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\left(Z_\epsilon^{\leq}-\frac{\beta^3}{3!}\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right)\right). \tag{3.70}$$

In order to prove that the first term on the r.h.s. of (3.68) vanishes, it thus remains to prove that

Lemma 3.15. *For all $\beta < \beta_p$,*

$$\lim_{N\uparrow\infty} N^{\left(\frac{3p}{2}-3\right)}\left|\mathbb{E}\left((Z_\epsilon^{\leq})^2\right)-\left(\mathbb{E}\left(Z_\epsilon^{\leq}\right)\right)^2-\frac{\beta^6}{3!^2}\mathbb{E}\left(\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)^2\right)\right| = 0 \tag{3.71}$$

and

Lemma 3.16. *For all $\beta \in \mathbb{R}_+$,*

$$\lim_{N\uparrow\infty} N^{\left(\frac{3p}{2}-3\right)}\left|\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\left(Z_\epsilon^{\leq}-\frac{\beta^3}{3!}\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right)\right)\right| = 0. \tag{3.72}$$

Lemma 3.15 and 3.16 clearly imply Proposition 3.2 for p even.

Proof of Lemma 3.15. We will now improve the estimate of the second moment of $Z_\epsilon^{\leq}$ started with Eq. (2.29). We write

$$\mathbb{E}\left[(Z_\epsilon^{\leq})^2\right] = A + B, \tag{3.73}$$

with A, B are given in (2.29) and $A \leq A_1 + A_2$, with A_1, A_2 defined in (2.48) and (2.49). The estimates obtained in Section 2 for B (Lemma 2.5) and A_1 (Eq. (2.58)) are good enough, but we need to improve the bound on A_2. Recall that in the final bound (2.67) for A_1 there was an error term of order $N^{3-3p/2}$, which would not vanish if multiplied with the $N^{3p/2-3}$. This term is due to the cubic term in the expansion (2.62). But this term reads

$$\frac{1}{3!}\sum_{m\in\Gamma_N}\left(\frac{\beta^2 N f_N^p(m)}{2\beta^2 a_N^2+1}\right)^3 p_N(m). \tag{3.74}$$

But recall that

$$\mathbb{E}\left(\frac{\beta^6}{3!^2}\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)^2\right) = \sum_{m\in\Gamma_N}\frac{\left(\beta^2 N f_N^p(m)\right)^3}{3!}p_N(m). \tag{3.75}$$

Therefore, im the expression in (3.71), this term exactly cancels the unpleasant cubic term in the expansion of A_2.

Recalling (2.63),

$$A_2 = \sum_{m\in\Gamma_N}\left(1+\frac{1}{2}\left(\frac{\beta^2 N f_N^p(m)}{2\beta^2 a_N^2+1}\right)^2+\frac{1}{3!}\left(\frac{\beta^2 N f_N^p(m)}{2\beta^2 a_N^2+1}\right)^3\right)p_N(m)+O\left(N^{4-2p}\right). \tag{3.76}$$

Hence, we arrive at

$$\begin{aligned}\mathbb{E}\left(Z_\epsilon^{\leq 2}\right) &= \left(1+\frac{\beta^4 N a_N^2}{2}+\frac{1}{3!}\sum_{m\in\Gamma_N}\left(\beta^2 N f_N^p(m)\right)^3 p_N(m)\right)\left(1-\beta^4 N a_N^2+O(N^{4-2p})\right)\\ &= 1-\frac{\beta^4 N a_N^2}{2}+\frac{1}{3!}\sum_{m\in\Gamma_N}\left(\beta^2 N f_N^p(m)\right)^3 p_N(m)+O(N^{4-2p}).\end{aligned} \tag{3.77}$$

By (2.76),

$$\mathbb{E}(Z_\epsilon^{\leq})^2 = 1 - \frac{\beta^4 N a_N^2}{2} + O\left(N^{4-2p}\right), \tag{3.78}$$

finally using (3.75), we get Lemma 3.15 and the lemma is proven. □

Proof of Lemma 3.16. By definition of $Z_\epsilon^{\leq}$, we can re-write

$$\begin{aligned}
&\left|\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\left(Z_\epsilon^{\leq} - \frac{\beta^3}{3!}\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right)\right)\right| \qquad (3.79)\\
&= \left|\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)(\mathcal{Z}_N(\beta) - Z_\epsilon^{>})\right) - \frac{\beta^3}{3!}\mathbb{E}\left(\left(\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)\right)^2\right)\right|\\
&\leq \left|\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\mathcal{Z}_N(\beta)\right) - \frac{\beta^3}{3!}\mathbb{E}\left(\left(\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)\right)^2\right)\right| + \left|\mathbb{E}\left(\mathbb{E}_\sigma\left(H_N(\sigma)^3\right)Z_\epsilon^{>}\right)\right|.
\end{aligned}$$

The first term of the last line can be calculated explicitly. By (3.13),

$$\begin{aligned}
\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\mathcal{Z}_N(\beta)\right) &= a_N^3 \sum_{(\neq)} \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\right)\mathbb{E}\left(J_A J_B J_C \mathcal{Z}_N(\beta)\right)\\
&= a_N^3 \sum_{(\neq)} \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\right)\mathbb{E}_{\sigma'}\mathbb{E}\left(J_A J_B J_C \mathrm{e}^{-\beta H_N(\sigma')-NJ_N(\beta)}\right).
\end{aligned} \tag{3.80}$$

Now,

$$\begin{aligned}
\mathbb{E}\left(J_A J_B J_C \mathrm{e}^{-\beta H_N(\sigma)-NJ_N(\beta)}\right) &= \mathbb{E}\left(J_A J_B J_C \mathrm{e}^{\sum_{D\in I_N}\left(\beta a_N\sigma_D J_D - \frac{\beta^2 a_N^2}{2}J_D^2\right)}\right)\\
&= \prod_{D\,\in I\setminus\{A,B,C\}} \mathbb{E}\left(\mathrm{e}^{\beta a_N\sigma_D J_D - \frac{\beta^2 a_N^2}{2}J_D^2}\right) \prod_{D\,\in\{A,B,C\}} \mathbb{E}\left(J_D \mathrm{e}^{a_N\beta\sigma_D J_D - \frac{a_N^2\beta^2}{2}J_D^2}\right).
\end{aligned} \tag{3.81}$$

We already have computed the terms in the first product, see (2.16). For the second, we get by elementary integration,

$$\mathbb{E}\left(J_D \mathrm{e}^{\beta a_N\sigma_D J_D - \frac{1}{2}\beta^2 a_N^2 J_D^2}\right) = \mathrm{e}^{\left(\frac{\beta^2 a_N^2}{2(1+2\beta^2 a_N^2)} - \frac{1}{2}\ln\left(1+\beta^2 a_N^2\right)\right)} \frac{\beta a_N \sigma_D}{\left(1+\beta^2 a_N^2\right)}. \tag{3.82}$$

Therefore,

$$\mathbb{E}\left(J_A J_B J_C \mathrm{e}^{H-NJ_N(\beta)}\right) = \mathrm{e}^{\binom{N}{p}\left(\frac{\beta^2 a_N^2}{2(1+2\beta^2 a_N^2)} - \frac{1}{2}\ln\left(1+\beta^2 a_N^2\right)\right)} \frac{\beta^3 a_N^3 \sigma_A\sigma_B\sigma_C}{\left(1+\beta^2 a_N^2\right)^3}. \tag{3.83}$$

Using (3.83) in (3.80) gives that

$$\begin{aligned}
\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\mathcal{Z}_N(\beta)\right) &= \frac{\beta^3 a_N^6}{\left(1+\beta^2 a_N^2\right)^3} \mathrm{e}^{\binom{N}{p}\left(\frac{\beta^2 a_N^2}{2(1+2\beta^2 a_N^2)} - \frac{1}{2}\ln\left(1+\beta^2 a_N^2\right)\right)} \sum_{(\neq)} \mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\right)^2\\
&= \frac{\beta^3 \mathbb{E}\left(\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right)^2\right)}{3!\left(1+\beta^2 a_N^2\right)^3} \mathrm{e}^{\binom{N}{p}\left(\frac{\beta^2 a_N^2}{2(1+2\beta^2 a_N^2)} - \frac{1}{2}\ln\left(1+\beta^2 a_N^2\right)\right)}.
\end{aligned} \tag{3.84}$$

Using (3.84), we get

$$\left|\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\mathcal{Z}_N(\beta)\right)-\frac{\beta^3}{3!}\mathbb{E}\left(\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right)^2\right)\right|$$
$$=\frac{\beta^3}{3!}\mathbb{E}\left(\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right)^2\right)\left(\frac{\mathrm{e}^{\binom{N}{p}\left(\frac{\beta^2a_N^2}{2(1+2\beta^2a_N^2)}-\frac{1}{2}\ln(1+\beta^2a_N^2)\right)}}{\left(1+\beta^2a_N^2\right)^3}-1\right). \tag{3.85}$$

A simple expansion shows that

$$\frac{\mathrm{e}^{\binom{N}{p}\left(\frac{\beta^2a_N^2}{2(1+2\beta^2a_N^2)}-\frac{1}{2}\ln(1+\beta^2a_N^2)\right)}}{\left(1+\beta^2a_N^2\right)^3}-1=O(N^{2-p}). \tag{3.86}$$

Since

$$N^{\frac{3p}{2}-3}\frac{\beta^6}{3!^2}\mathbb{E}\left(\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right)^2\right)\to\sigma(\beta,p)^2, \tag{3.87}$$

and therefore

$$\lim_{N\to\infty}N^{\left(\frac{3p}{2}-3\right)}\left|\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\mathcal{Z}_N(\beta)\right)-\frac{\beta^3}{3!}\mathbb{E}\left(\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right)^2\right)\right|=0. \tag{3.88}$$

It remains to prove that $\left|\mathbb{E}\left(\frac{\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)}{3!}Z_\epsilon^>\right)\right|$ tends to 0. To see that, we use the Hölder inequality.

$$\left|\mathbb{E}\left(\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)Z_\epsilon^>\right)\right|\le\mathbb{E}\left(\left|\mathbb{E}_{\sigma'}\left(-H_N(\sigma)^3\right)\right|\mathbb{E}_\sigma\left(\mathrm{e}^{-\beta H_N(\sigma)}\mathbb{1}_{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\mathrm{e}^{-NJ_N(\beta)}\right)\right)$$
$$=\mathbb{E}_\sigma\left(\mathbb{E}\left(\left|\mathbb{E}_{\sigma'}\left(-H_N(\sigma)^3\right)\right|\mathrm{e}^{-\beta H_N(\sigma)}\mathbb{1}_{|-H_N(\sigma)-\beta N|>\epsilon\beta N\}}\mathrm{e}^{-NJ_N(\beta)}\right)\right)$$
$$\le\mathbb{E}_\sigma\left(\mathbb{E}\left(\mathrm{e}^{q_1\beta\sqrt{N}X_\sigma}\mathbb{1}_{|X_\sigma-\beta\sqrt{N}|>\epsilon\beta\sqrt{N}\}}\right)^{\frac{1}{q_1}}\mathbb{E}\left(\left|\mathbb{E}_{\sigma'}\left(H_N(\sigma)^3\right)\right|^{q_2}\mathrm{e}^{-q_2NJ_N(\beta)}\right)^{\frac{1}{q_2}}\right), \tag{3.89}$$

for $\frac{1}{q_1}+\frac{1}{q_2}=1$. For the last factor, the Cauchy-Schwarz inequality gives

$$\mathbb{E}\left(\left|\mathbb{E}_{\sigma'}\left(H_N(\sigma)^3\right)\right|^{q_2}\mathrm{e}^{-q_2NJ_N(\beta)}\right)^{\frac{1}{q_2}}\le\mathbb{E}\left(\left|\mathbb{E}_{\sigma'}\left(-H_N(\sigma)^3\right)\right|^{2q_2}\right)^{\frac{1}{2q_2}}\mathbb{E}\left(\mathrm{e}^{-2q_2NJ_N(\beta)}\right)^{\frac{1}{2q_2}}. \tag{3.90}$$

Again by **Fact I** in the appendix, the last line in (3.89) is bounded from above by

$$\mathrm{e}^{\left(-\frac{(1+\epsilon)^2\beta^2N}{2q_1}+(1+\epsilon)\beta^2N\right)}\left(\mathbb{E}\left(\left|\mathbb{E}_\sigma\left(-H_N(\sigma)^3\right)\right|^{2q_2}\right)^{\frac{1}{2q_2}}\mathbb{E}\left(\mathrm{e}^{-2q_2NJ_N(\beta)}\right)^{\frac{1}{2q_2}}\right). \tag{3.91}$$

Finally, by explicit computation,

$$\mathbb{E}\left(\mathrm{e}^{-2q_2NJ_N(\beta)}\right)^{1/2q_2}=\mathrm{e}^{-N\beta^2/2+O\left(N^{2-p}\right)}. \tag{3.92}$$

Combining (3.91) and (3.92), we obtain

$$\left|\mathbb{E}\left(\mathbb{E}_{\sigma'}\left(H_N(\sigma)^3\right)Z_\epsilon^>\right)\right|$$
$$\le\exp\left(-\beta^2N\left(\frac{\epsilon^2}{2}+O\left(q_1-1\right)+O\left(N^{2-p}\right)\right)\right)\mathbb{E}\left(\left|\mathbb{E}_{\sigma'}\left(-H_N(\sigma)^3\right)\right|^{2q_2}\right)^{\frac{1}{2q_2}}. \tag{3.93}$$

For every $\epsilon > 0$, we can choose q_1 close to 1 such that the first term on the r.h.s. of (3.93) is exponentially small. The second term will however stay polynomial. This concludes the proof of Lemma 3.16. □

This concludes the proof of Proposition 3.2 in case of p even.

□

3.5. **Proof of Proposition 3.2: p odd.** The proof in the odd case is in principle similar to the even case. It is enough to show that

$$\lim_{N\uparrow\infty} N^{p-2}\left(Z_\epsilon^{\leq} - T_N(\beta)\right) = 0, \tag{3.94}$$

in probability. Using (3.19) we decompose

$$\left|Z_\epsilon^{\leq} - T_N(\beta)\right| \leq \left|Z_\epsilon^{\leq} - \beta^4\mathcal{H}_4 - \mathbb{E}\left(Z_\epsilon^{\leq}\right)\right| + \left|\mathbb{E}\left(Z_\epsilon^{\leq}\right) - 1 + \frac{2\beta^4 a_N^4}{4!}\sum_{A\in I} J_A^4\right|. \tag{3.95}$$

The second term is irrelevant. Using (2.24) and the law of large numbers from (3.21), we see that the second term is smaller than $o(N^{2-p})$ and hence gives a vanishing contribution to (3.94). For the first term in (3.95) we control its second moment. We write

$$\begin{aligned}\mathbb{E}\left(\left(Z_\epsilon^{\leq} - \beta^4\mathcal{H}_4 - \mathbb{E}\left(Z_\epsilon^{\leq}\right)\right)^2\right) &= 2\beta^4\mathbb{E}\left(\mathcal{H}_4\mathbb{E}\left(Z_\epsilon^{\leq}\right)\right) - 2\beta^4\mathbb{E}\left(\mathcal{H}_4 Z_\epsilon^{\leq}\right) + \mathbb{E}\left({Z_\epsilon^{\leq}}^2\right) - \mathbb{E}\left(Z_\epsilon^{\leq}\right)^2 \\ &\quad +\beta^8\mathbb{E}\left({\mathcal{H}_4}^2\right) \qquad (3.96)\\ &= 2\beta^4\mathbb{E}\left(\mathcal{H}_4(\beta^4\mathcal{H}_4 - Z_\epsilon^{\leq})\right)\right) + \mathbb{E}\left({Z_\epsilon^{\leq}}^2\right) - \mathbb{E}\left(Z_\epsilon^{\leq}\right)^2 - \beta^8 E\left(\mathcal{H}_4^2\right),\end{aligned}$$

where we used that $\mathcal{H}_4$ has zero mean.

We will prove the following two lemmata.

Lemma 3.17. *For all β,*

$$\lim_{N\to+\infty} N^{2p-4}\left|\mathbb{E}\left(\mathcal{H}_4(Z_\epsilon^{\leq} - \beta^4\mathcal{H}_4)\right)\right| = 0. \tag{3.97}$$

and

Lemma 3.18. *For all $\beta < \beta_p$,*

$$\lim_{N\to+\infty} N^{2p-4}\left|\mathbb{E}\left({Z_\epsilon^{\leq}}^2\right) - \mathbb{E}\left(Z_\epsilon^{\leq}\right)^2 - \beta^8\mathbb{E}\left(\mathcal{H}_4^2\right)\right| = 0. \tag{3.98}$$

We will first prove Lemma 3.17 by following exactly the same strategy as for the case p even.

Proof of Lemma 3.17. The proof of this lemma is very similar to that of Lemma 3.16 and we omit many details. As in (3.79), we start with

$$\left|\mathbb{E}\left(\mathcal{H}_4(Z_\epsilon^{\leq} - \beta^4\mathcal{H}_4)\right)\right| \leq \left|\mathbb{E}\left(\mathcal{H}_4\mathcal{Z}_N(\beta)\right) - \beta^4\mathbb{E}\left(\mathcal{H}_4^2\right)\right| + \left|\mathbb{E}\left(\mathcal{H}_4 Z_\epsilon^{>}\right)\right|. \tag{3.99}$$

For the first term on the r.h.s. of (3.99), we have

$$\mathbb{E}\left(\mathcal{H}_4\mathcal{Z}_N(\beta)\right) = \frac{a_N^4}{4!}\sum_{(\neq)}\mathbb{E}_\sigma\left(\sigma_A\sigma_B\sigma_C\sigma_D\right)\mathbb{E}_\sigma\mathbb{E}\left(J_A J_B J_C J_D \mathrm{e}^{-H_N(s)-NJ_N(\beta)}\right). \tag{3.100}$$

Following now the exact same steps as in the proof of 3.16, we arrive at the analog of (3.84),

$$\mathbb{E}\left(\mathcal{H}_4\mathcal{Z}_N(\beta)\right) = \frac{\beta^4\mathbb{E}\left(\mathcal{H}_4^2\right)}{\left(1+\beta^2 a_N^2\right)^4}\mathrm{e}^{\binom{N}{p}\left(\frac{\beta^2 a_N^2}{2(1+2\beta^2 a_N^2)} - \frac{1}{2}\ln(1+\beta^2 a_N^2)\right)}. \tag{3.101}$$

From here one concludes that

$$\lim_{N\uparrow\infty} N^{2p-4}\left|\mathbb{E}\left(\mathcal{H}_4\mathcal{Z}_N(\beta)\right)-\beta^4\mathbb{E}\left(\mathcal{H}_4^2\right)\right| = 0. \tag{3.102}$$

The second term on the right of (3.99) is shown to be exponentially small exactly as the second term in (3.79). This concludes the proof of Lemma 3.17. □

Proof of Lemma 3.18. It remains to prove that

$$\lim_{N\to+\infty} N^{2p-4}\left|\mathbb{E}\left(Z_\epsilon^{\leq 2}\right)-\mathbb{E}\left(Z_\epsilon^{\leq}\right)^2-\beta^8\mathbb{E}\left(\mathcal{H}_4^2\right)\right| = 0. \tag{3.103}$$

As in the proof of Lemma 3.15, we improve the estimate on $\mathbb{E}\left((Z_\epsilon^{\leq})^2\right)$ by retaining an additional term in the expansion of the exponential that then is cancelled by the $\mathbb{E}\left(\mathcal{H}_4^2\right)$. Again this involves only the term A_2. This time, this requires to push the expansion further and to use that

$$\left|\exp(\xi)-1-\xi-\frac{1}{2}\xi^2-\frac{1}{3!}\xi^3-\frac{1}{4!}\xi^4-\frac{1}{5!}\xi^5\right| \leq \frac{1}{6!}\xi^6\exp|\xi|. \tag{3.104}$$

This leads to the estimate

$$\begin{aligned}
&\mathbb{E}\left(Z_\epsilon^{\leq 2}\right) \qquad (3.105)\\
&= \sum_{m\in\Gamma_N}\left(1+\frac{1}{2}\left(\frac{\beta^2Nf_N^p(m)}{2a_N^2+1}\right)^2+\frac{1}{4!}\left(\frac{\beta^2Nf_N^p(m)}{2a_N^2+1}\right)^4\right)e^{-\beta^4Na_N^2+O\left(N^{3-2p}\right)}+o(N^{4-2p}),
\end{aligned}$$

where we used that the terms of odd order vanish by symmetry. The quadratic term equals $\frac{\beta^4N^2}{2\binom{N}{p}(2\beta^2a_N^2+1)^2}$. Moreover, the quartic term gives

$$\frac{1}{4!}\sum_{m\in\Gamma_N} p_N(m)\left(\frac{\beta^2Nf_N^p(m)}{2a_N^2+1}\right)^4 = \beta^8\mathbb{E}\left(\mathcal{H}_4^2\right)+\frac{\beta^8N^2a_N^4}{8}+O\left(N^{4-3p}\right). \tag{3.106}$$

Furthermore, using (2.13) we have that

$$\begin{aligned}
\mathbb{E}(Z_\epsilon^{\leq})^2 &= \left(1-\frac{\beta^4}{4}Na_N^2+\frac{\beta^8}{32}N^2a_N^4+O\left(N^{3-2p}\right)\right)^2\\
&= 1-\frac{\beta^4Na_N^2}{2}+\frac{2\beta^8N^2a_N^4}{16}+O\left(N^{3-2p}\right).
\end{aligned} \tag{3.107}$$

Combining these observations, the assertion of Lemma 3.18 follows. □

This concludes the proof of Proposition 3.2 and hence of Theorem 1.2.

4. Appendix

We state three useful results for the convenience of the reader. The first concerns standard estimates for truncated exponential moments of Gaussian random variables.

Fact I. *Let ξ be a Gaussian random variable with $\mathbb{E}(\xi) = 0$, $\mathbb{E}(\xi^2) = 1$. Then for all $a, b > 0$*

$$\mathbb{E}[e^{a\xi}\mathbb{1}_{\{\xi>b\}}] \leq \frac{1}{\sqrt{2\pi}(b-a)}e^{-b^2/2+ab}, \quad \textit{if } b > a, \tag{4.1}$$

$$\mathbb{E}[e^{a\xi}\mathbb{1}_{\{\xi<b\}}] \leq \frac{1}{\sqrt{2\pi}(a-b)}e^{-b^2/2+ab}, \quad \textit{if } b < a. \tag{4.2}$$

The second is the Gaussian concentration of measure inequality, to be found, for example, in [11].

Fact II. *Assume that $f(x_1, \ldots, x_d)$ is a function on $\mathbb{R}^d$ with a Lipschitz constant L. Let $J_1, \ldots, J_d$ be independent standard Gaussian random variables. Then for any $u > 0$*

$$\mathbb{P}\{|f(J_1, \ldots, J_d) - \mathbb{E}\,(f(J_1, \ldots, J_d))\,| > u\} \leq 2\exp\{-u^2/(2L^2)\}. \tag{4.3}$$

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