The Two Point Function of the SK Model without External Field at High Temperature

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Abstract

We show that the two point correlation matrix $\mathbf{M} = (\langle\sigma_i\sigma_j\rangle)_{1\leq i,j\leq N}$ of the Sherrington-Kirkpatrick model with zero external field satisfies

$$\lim_{N\to\infty} \|\mathbf{M} - (1+\beta^2-\beta\mathbf{G})^{-1}\|_{\mathrm{op}} = 0$$

in probability, in the full high temperature regime $\beta < 1$. Here, $\mathbf{G}$ denotes the GOE interaction matrix of the model.

1 Introduction

Consider N spins σ_i, $i \in \{1,\ldots,N\}$, with values in $\{-1,1\}$ whose interactions are described by the Sherrington-Kirkpatrick [11] Hamiltonian $H_N : \{-1,1\}^N \to \mathbb{R}$

$$H_N(\sigma) = \beta \sum_{1\leq i<j\leq N} g_{ij}\sigma_i\sigma_j = \frac{\beta}{2}(\sigma, \mathbf{G}\sigma). \tag{1.1}$$

The symmetric interaction $\mathbf{G} = (g_{ij})_{1\leq i,j\leq N}$ is a GOE matrix, i.e. its entries are i.i.d. centered Gaussian random variables of variance N^{-1} for $i \neq j$ (we set $g_{ii} := 0$ for simplicity), and $\beta \geq 0$ denotes the inverse temperature. We assume the $\{g_{ij}\}$ to be realized in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote the expectation with respect to them by $\mathbb{E}(\cdot)$. We denote the $L^p(\Omega,\mathcal{F},\mathbb{P})$ norms by $\|\cdot\|_{L^p(\Omega)} = (\mathbb{E}\,|\cdot|^p)^{1/p}$, for $p \geq 1$.

In this paper, we are interested in analyzing the behavior of the two point correlation matrix of the model at high temperature. The correlation matrix is given by

$$\mathbf{M} = \big(\langle\sigma_i\sigma_j\rangle\big)_{1\leq i,j\leq N} \in \mathbb{R}^{N\times N}, \tag{1.2}$$

where $\langle\cdot\rangle$ denotes from now on the Gibbs expectation which is defined so that

$$\langle f\rangle = \frac{1}{Z_N}\frac{1}{2^N}\sum_{\sigma\in\{-1,1\}^N} f(\sigma)\, e^{H_N(\sigma)} \quad \text{with} \quad Z_N = \frac{1}{2^N}\sum_{\sigma\in\{-1,1\}^N} e^{H_N(\sigma)} \tag{1.3}$$

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for every observable $f : \{-1,1\}^N \to \mathbb{R}$.

As discussed recently in [1] (see in particular [1, Corollary 1.4 & Remark 1.5] and the references therein), standard mean field heuristics suggests that, at sufficiently high temperature, the two point correlations satisfy the self-consistent equations[1]

$$\langle \sigma_i \sigma_j \rangle \approx \sum_{k=1}^{N} \beta g_{ik} \langle \sigma_k \sigma_j \rangle - \beta^2 \langle \sigma_i \sigma_j \rangle. \tag{1.4}$$

This is analogous to the well-known Thouless-Anderson-Palmer equations [13] for the magnetizations at high temperature and non-zero external field. By (1.4), we thus expect

$$\mathbf{M} \approx \frac{1}{1+\beta^2 - \beta \mathbf{G}} \tag{1.5}$$

in a suitable sense to be made precise. Since the spectrum of a GOE matrix $\mathbf{G}$ is contained with high probability in $[-2;2]$, the simple mean field heuristics naturally suggests the (well-known) phase transition at $\beta = 1$, without referring to replica arguments or the powerful Parisi theory [9, 10]. This is our main motivation and giving a rigorous proof of (1.5) for all $\beta < 1$ is our main result. In the sequel, we denote by $\|\cdot\|$ the standard Euclidean norm in $\mathbb{R}^N$ while $\|\mathbf{A}\|_{\text{op}}$ and $\|\mathbf{A}\|_{\text{F}}$, for $\mathbf{A} = (a_{ij})_{1\leq i,j\leq N} \in \mathbb{R}^{N\times N}$, refer to the operator and Frobenius norms, respectively, which are defined by

$$\|\mathbf{A}\|_{\text{op}} := \sup_{v\in\mathbb{R}^N : \|v\|=1} \|\mathbf{A}v\| \qquad \text{and} \qquad \|\mathbf{A}\|_{\text{F}} := \Big(\sum_{1\leq i,j\leq N} a_{ij}^2 \Big)^{1/2}.$$

Theorem 1.1. *Assume that $\beta < 1$ and denote by $\boldsymbol{P} = (p_{ij})_{1\leq i,j\leq N} \in \mathbb{R}^{N\times N}$ the matrix whose diagonal entries are equal to one and whose off-diagonal entries are given by*

$$p_{ij} = \sum_{\substack{\gamma:\ \textit{self-avoiding} \\ \textit{path from } i \textit{ to } j}} \prod_{i_k \in \gamma} \beta g_{i_k i_{k+1}}$$

for $i \neq j$ (see Eq. (2.25) for a precise definition). Then it holds true that

$$\lim_{N\to\infty} \|\boldsymbol{M} - \boldsymbol{P}\|_{\text{F}} = 0 \qquad \textit{and} \qquad \lim_{N\to\infty} \|\boldsymbol{P} - (1+\beta^2 - \beta \boldsymbol{G})^{-1}\|_{\text{op}} = 0$$

in the sense of probability. In particular, we have that

$$\lim_{N\to\infty} \|\boldsymbol{M} - (1+\beta^2 - \beta \boldsymbol{G})^{-1}\|_{\text{op}} = 0. \tag{1.6}$$

Remarks:

1) As a corollary of Theorem 1.1, it follows that, with high probability, the norm of $\mathbf{M}$ is given by $(1-\beta)^{-2}$ for large N, up to to an arbitrarily small error. Indeed, this corresponds to $\|(1+\beta^2 - \beta\mathbf{G})^{-1}\|_{\text{op}}$ (see, for instance, [4, Theorem C]).

[1] Notice that $\langle \sigma_i \rangle = 0$ for all $i \in \{1,\dots,N\}$ by the global spin-flip symmetry of the model.

2) The boundedness of $\mathbb{E}\|\mathbf{M}\|_{\mathrm{op}}$ has been conjectured in [12, Conjecture 11.5.1], based on the simple observation that $m_{ij} \approx \beta g_{ij}$ to leading order in β (for $i \neq j$). While writing up our manuscript, we became aware that this has been verified quite recently in an independent work [8], combining the TAP heuristics (1.4) with well established higher-order asymptotics for overlap moments [12, 5]. The main result of [8] shows that $\mathbb{E}\|\mathbf{M}\,(1+\beta^2-\beta\mathbf{G})-\mathrm{id}_{\mathbb{R}^N}\|_{\mathrm{F}} = O(1)$, implying $\mathbb{E}\|\mathbf{M}\|_{\mathrm{op}} = O(1)$. In particular, the matrix elements of $\mathbf{M}\,(1+\beta^2-\beta\mathbf{G})-\mathrm{id}_{\mathbb{R}^N}$ are typically of size $O(N^{-1})$. This information is, however, not enough to conclude the norm convergence (1.6) in Theorem 1.1. Inspired by [2], our proof of (1.6) is based on a microcanonical analysis that determines the entries of $\mathbf{M}$ explicitly in terms of $\mathbf{G}$. This enables us not only to determine the typical size of the entries of $\mathbf{M}\,(1+\beta^2-\beta\mathbf{G})-\mathrm{id}_{\mathbb{R}^N}$, but also to conclude that its operator norm vanishes. In particular, our arguments are independent of [12, 5, 8].

3) The recent paper [6] establishes a logarithmic Sobolev inequality for the Gibbs measure induced by H_N under the stronger high temperature condition $\beta < 1/4$. This implies in particular the boundedness of the norm of $\mathbf{M}$. Whether the results of [6] or, possibly simpler, a spectral gap inequality can be proved for all $\beta < 1$ remains an interesting open question. Theorem 1.1 may be viewed as an initial step in this direction as it implies a spectral gap inequality on the space of linear combinations of the magnetizations σ_i.

4) Theorem 1.1 is trivial if $\beta = 0$. We therefore assume from now on that $\beta > 0$.

Let us briefly outline the strategy of our proof. As shown in [2], which studies the fluctuations $\Phi = \log Z_N - \frac{\beta^2}{4}N$ of the log partition function around its leading order contribution, one may think of $\log Z_N$ for $\beta < 1$ heuristically as

$$\begin{aligned}\log Z_N &= \sum_{1\leq i<j\leq N} \log\cosh(\beta g_{ij}) + \log\frac{1}{2^N}\sum_{\sigma\in\{-1,1\}^N}\prod_{1\leq i<j\leq N}\big(1+\tanh(\beta g_{ij})\sigma_i\sigma_j\big)\\ &\approx \sum_{1\leq i<j\leq N} \log\cosh(\beta g_{ij}) + \sum_{\gamma\text{ simple loop}} \log(1+w(\gamma)),\end{aligned} \tag{1.7}$$

where by simple loops, we mean simple, connected graphs γ with vertices in $\{1,\dots,N\}$ each having degree two (see section 2 for the details) and with corresponding weights

$$w(\gamma) = \prod_{\{i,j\}\in\gamma} \tanh(\beta g_{ij}).$$

As a consequence of (1.7) we expect that up to negligible errors, we have for $i \neq j$

$$\begin{aligned}\langle\sigma_i\sigma_j\rangle = \beta^{-1}\partial_{ij}\log Z_N &\approx \tanh(\beta g_{ij}) + \sum_{\{i,j\}\in\gamma\text{ simple loop}} \frac{w(\gamma)}{\tanh(\beta g_{ij})}\frac{1}{1+w(\gamma)}\\ &\approx \beta g_{ij} + \sum_{\{i,j\}\in\gamma\text{ simple loop}}\ \prod_{e\in\gamma: e\neq\{i,j\}} \beta g_e.\end{aligned} \tag{1.8}$$

In other words, the correlation between spins σ_i and σ_j can be written as the sum over weights of self-avoiding paths from vertex i to vertex j.

On the other hand, by standard mean field arguments, we expect that

$$\big(\mathbf{M}(1+\beta^2-\beta\mathbf{G})\big)_{ij} \approx \delta_{ij}.$$

That this is consistent with (1.8) can readily be checked on a heuristic level. Consider for instance the diagonal terms for which one obtains

$$\begin{aligned}\big(\mathbf{M}(1+\beta^2-\beta\mathbf{G}) - \mathrm{id}_{\mathbb{R}^N}\big)_{ii} &\approx \beta^2\bigg(1-\sum_{k\neq i} g_{ik}^2\bigg) - \sum_{k\neq i}\ \sum_{\{i,k\}\in\gamma \text{ simple loop}} \beta g_{ik} \prod_{e\in\gamma: e\neq\{i,k\}} \beta g_e \\ &\approx \beta^2\bigg(1-\sum_{k\neq i} g_{ik}^2\bigg) - 2 \sum_{\gamma \text{ simple loop}: \, i\in\mathcal{V}_\gamma}\ \prod_{e\in\mathcal{E}_\gamma} \beta g_e = O(N^{-1/2})\end{aligned}$$

with high probability, uniformly in $i\in\{1,\dots,N\}$. Here $\mathcal{V}_\gamma$ and $\mathcal{E}_\gamma$ denote the vertex and edge sets of γ. Notice that all order one contributions are cancelled. Similar cancellations suggest the smallness of the off-diagonal elements of $\mathbf{M}(1+\beta^2-\beta\mathbf{G})$. That is, we find

$$\big(\mathbf{M}(1+\beta^2-\beta\mathbf{G})\big)_{ij} = O(N^{-1})$$

for $i\neq j$, with high probability. This is enough to bound $\|\mathbf{M}\|_{\mathrm{op}}$ uniformly in N, but it does not, yet, allow us to conclude that

$$\|\mathbf{M}(1+\beta^2-\beta\mathbf{G}) - \mathrm{id}_{\mathbb{R}^N}\|_{\mathrm{op}} \to 0$$

as $N\to\infty$. To show this, we prove in fact an identity of the form

$$\mathbf{M}(1+\beta^2-\beta\mathbf{G}) - \mathrm{id}_{\mathbb{R}^N} = \big(\mathbf{M}(1+\beta^2-\beta\mathbf{G}) - \mathrm{id}_{\mathbb{R}^N}\big)\mathbf{Q}' + \mathbf{Q}''$$

for two errors $\mathbf{Q}', \mathbf{Q}''$ which have vanishing operator norm in the limit $N\to\infty$ if $\beta<1$ (see Corollary 2.9 and Eq. (3.8) for the details), which clearly implies (1.6).

To make the above steps rigorous, our main task is to verify the approximation (1.8) in a sufficiently strong sense. From the technical point of view, this is our main contribution, thereby extending the high temperature analysis of [2] and providing a direct explanation of the emergence of the resolvent of $\mathbf{G}$.

The paper is structured as follows. In Section 2 we set up the notation, we collect several preliminary results and we explain the reduction step from $\mathbf{M}$ to $\mathbf{P}$. Using this information, we conclude Theorem 1.1 in Section 3 by comparing the matrix $\mathbf{P}$ with the resolvent $(1+\beta^2-\beta\mathbf{G})^{-1}$.

2 Setup and Reduction Step

In this section, we set up the notation following [2], we collect several preliminary results and we determine the main contribution $\mathbf{P}$ to the two point correlation matrix $\mathbf{M}$.

We start with the graphical representation of the log partition function

$$\begin{aligned}\log Z_N &= \sum_{1\le u<v\le N} \log\cosh(\beta g_{uv}) + \log \Big\langle \prod_{1\le u<v\le N} (1+\tanh(\beta g_{uv})\sigma_u\sigma_v)\Big\rangle_0 \\ &= \sum_{1\le u<v\le N} \log\cosh(\beta g_{uv}) + \log \sum_{\gamma\in\Gamma_{\mathrm{sc}}} w(\gamma) \\ &=: \sum_{1\le u<v\le N} \log\cosh(\beta g_{uv}) + \log \widehat{Z}_N\end{aligned} \tag{2.1}$$

and, analogously, of the two point correlation functions

$$\langle\sigma_i\sigma_j\rangle = \beta^{-1}\partial_{ij}\log Z_N = \tanh(\beta g_{ij}) + (1-\tanh^2(\beta g_{ij}))\, \widehat{Z}_N^{-1} \sum_{\substack{\gamma\in\Gamma_{\mathrm{sc}}:\\ \{i,j\}\in\gamma}} \frac{w(\gamma)}{\tanh(\beta g_{ij})} \tag{2.2}$$

for $i\neq j$ (recall that $\langle\sigma_i\sigma_i\rangle = 1$), where

$$w(\gamma) = \prod_{e\in\mathcal{E}_\gamma} \tanh(\beta g_e). \tag{2.3}$$

Here and from now on, $\langle\cdot\rangle_0$ denotes the expectation with regards to the law of N i.i.d. Bernoulli variables σ_i, for $i\in\{1,\dots,N\}$. Notice that $\langle\cdot\rangle_0$ is obtained from $\langle\cdot\rangle$, defined in (1.3), by setting $\beta\equiv 0$ in (1.1).

$\Gamma_{\mathrm{sc}} = \Gamma_{\mathrm{sc}}([N])$ denotes the set of simple, closed graphs with vertices in $[N] := \{1,\dots,N\}$. To be more precise, $\gamma = (\mathcal{V}_\gamma,\mathcal{E}_\gamma)\in\Gamma_{\mathrm{s}} = \Gamma_{\mathrm{s}}[N]$ is called simple for subsets $\mathcal{V}_\gamma\subset\{1,\dots,N\}$ (the vertex set of γ) and $\mathcal{E}_\gamma\subset\big\{\{i,j\} : i,j\in\mathcal{V}_\gamma,\, i\neq j\big\}$ (the set of edges of γ) if γ contains no isolated vertices (for every $i\in\mathcal{V}_\gamma$ there exists some $e\in\mathcal{E}_\gamma$ with $i\in e$) and if the multiplicity $n_{ij}(\gamma)\in\{0,1\}$ is at most one, for each edge $\{i,j\}$. Unless stated otherwise, we set $n_{ii}(\gamma)\equiv 0$ for all $i\in\{1,\dots,N\}$ (self-loops are not allowed). $\gamma\in\Gamma_{\mathrm{sc}}\subset\Gamma_{\mathrm{s}}$ is called a simple, closed graph if it is simple and if the degree

$$n_i(\gamma) = \sum_{j=1}^N n_{ij}(\gamma) \in 2\mathbb{N}_0$$

of each vertex $i\in\mathcal{V}_\gamma$ is even. By convention, we include $\emptyset\in\Gamma_{\mathrm{sc}}$ with weight $w(\emptyset) := 1$.

Besides Γ_{sc}, the set $\Gamma_{\mathrm{loop}}\subset\Gamma_{\mathrm{sc}}$ of simple loops (or, equivalently, cycles) will be of particular importance. It is defined as the set

$$\Gamma_{\mathrm{loop}} = \big\{\gamma\in\Gamma_{\mathrm{sc}} : n_i(\gamma) = 2\ \forall\, i\in\mathcal{V}_\gamma \text{ and } \gamma \text{ is connected}\big\}.$$

Recall that a graph is connected if for every pair $i,j\in\mathcal{V}_\gamma$ of vertices, γ contains a path from i to j. That is, there exist vertices $v_1,\dots,v_k\in\mathcal{V}_\gamma$ so that

$$\big(\{i,v_1\},\{v_1,v_2\},\dots,\{v_{k-1},v_k\},\{v_k,j\}\big)\in\mathcal{E}_\gamma^k.$$

We call a path self-avoiding if each of its vertices occurs at most as part of two edges. Moreover, recall from Veblen's theorem [7, Theorem 2.7] that every $\gamma \in \Gamma_{\text{sc}}$ is equal to an edge-disjoint union of a finite number of cycles $\gamma_1, \dots, \gamma_k \in \Gamma_{\text{loop}}$, which we write as

$$\gamma = \gamma_1 \circ \gamma_2 \circ \dots \circ \gamma_k. \tag{2.4}$$

If $e = \{i, j\} \in \mathcal{E}_\gamma$, we set $g_e = g_{ij}$, and from now on, by slight abuse of notation, we frequently write $e \in \gamma$ edges $e \in \mathcal{E}_\gamma$. Similarly, we write $|\gamma|$ $(= |\mathcal{E}_\gamma|)$ to denote the number of edges in γ and by $\gamma \cap \gamma' = \emptyset$, we abbreviate that two graphs γ, γ' are edge-disjoint (i.e. $\mathcal{E}_\gamma \cap \mathcal{E}_{\gamma'} = \emptyset$). Finally, constants independent of N are typically denoted by $C, C', c, c', etc.$ and they may vary from line to line in our estimates.

Let us now prepare the analysis of the two point function by collecting several preliminary results. We first recall one of the main results of [2] that tells us that the fluctuation term $\widehat{Z}_N$, defined in (2.1), converges in distribution to a log-normal variable.

Proposition 2.1. *([2, Prop. 2.2]) Let $\beta < 1$ and set*

$$\sigma^2 = \sum_{k \geq 3} \frac{\beta^{2k}}{2k}.$$

Then $\widehat{Z}_N$, defined in (2.1)*, converges in distribution to the log-normal variable*

$$\lim_{N \to \infty} \widehat{Z}_N \stackrel{\mathrm{d}}{=} \exp\Big(Y - \frac{\sigma^2}{2}\Big),$$

where $Y \sim \mathcal{N}(0, \sigma^2)$ denotes a centered, Gaussian random variable of variance σ^2.

A crucial observation from [2], used to derive Theorem 2.1, is that large graphs have exponentially decaying $L^2(\Omega)$ norm, if $\beta < 1$. In the sequel, we frequently use this result with mildly diverging large graph cutoff $(\log N)^{1+\epsilon} \to \infty$ as $N \to \infty$ for suitable $\epsilon > 0$.

Lemma 2.2. *([2, Lemma 3.3]) Let $\beta < 1$, then there exists a constant $C = C_\beta > 0$, which is independent of N, such that*

$$\mathbb{E}\Big(\sum_{\gamma \in \Gamma_{\text{sc}}: |\gamma| \geq k} w(\gamma)\Big)^2 \leq C \exp\big(-k \log \beta^{-1}\big)$$

for every $k \geq 0$. Similarly, for every $i, j \in [N]$ with $i \neq j$, it holds true that

$$\mathbb{E}\Big(\sum_{\substack{\gamma \in \Gamma_{\text{sc}}: \\ \{i,j\} \in \gamma, |\gamma| \geq k}} \frac{w(\gamma)}{\tanh(\beta g_{ij})}\Big)^2 \leq C \exp\big(-k \log \beta^{-1}\big).$$

Proof. For the reader's convenience, we recall the quick argument together with some useful facts about the weights $w(\gamma)$. We consider two cases.

First, if $k \le \max\big(\frac{1}{2}(1-\beta^2)^{-2}, (16)^2/(\log\beta^{-2})^2\big)$, we bound

$$\mathbb{E}\Big(\sum_{\gamma\in\Gamma_{\mathrm{sc}}:|\gamma|\ge k} w(\gamma)\Big)^2 = \sum_{\gamma\in\Gamma_{\mathrm{sc}}:|\gamma|\ge k} \mathbb{E}w^2(\gamma) \le \prod_{\gamma\in\Gamma_{\mathrm{loop}}} \big(1+\mathbb{E}w^2(\gamma)\big) \le \exp\Big(\sum_{j=3}^{\infty}\frac{\beta^{2j}}{2j}\Big)$$
$$\le \frac{1}{\sqrt{1-\beta^2}}.$$

In the first step, we used that the weights $w(\gamma_1)$ and $w(\gamma_2)$ of two different graphs $\gamma_1 \neq \gamma_2$ are orthogonal in $L^2(\Omega)$ and in the second step we used the observation from (2.4). Finally, in the third line we used the estimate (see [2, Eq. (3.12)] for the details)

$$\sum_{\gamma\in\Gamma_{\mathrm{loop}}:|\gamma|=j} \mathbb{E}w^2(\gamma) \le \frac{\beta^{2j}}{2j}.$$

For $k \le \max\big(\frac{1}{2}(1-\beta^2)^{-2}, (16)^2/(\log\beta^{-2})^2\big)$, we thus obtain trivially that

$$\mathbb{E}\Big(\sum_{\gamma\in\Gamma_{\mathrm{sc}}:|\gamma|\ge k} w(\gamma)\Big)^2 \le C'\exp\big(-k\log\beta^{-1}\big),$$

where $C' = (1-\beta^2)^{-1/2}\max\big(e^{(1-\beta^2)^{-2}\log\beta^{-1/2}}, e^{(16)^2/(4\log\beta^{-1})}\big)$.

On the other hand, assume $k > \max\big(\frac{1}{2}(1-\beta^2)^{-2}, (16)^2/(\log\beta^{-2})^2\big)$ s.t. in particular

$$\epsilon = \log\beta^{-2}(1-1/\sqrt{2k}) > 0.$$

In this case, we bound

$$\begin{aligned}\mathbb{E}\Big(\sum_{\gamma\in\Gamma_{\mathrm{sc}}:|\gamma|\ge k} w(\gamma)\Big)^2 &\le \sum_{l\ge1}\ \sum_{\substack{\{\gamma_1,\dots,\gamma_l\}:\gamma_i\in\Gamma_{\mathrm{loop}},\\ \gamma_i\neq\gamma_j,|\gamma_1|\ +\dots+|\gamma_l|\ge k}}\ \prod_{j=1}^{l}\mathbb{E}w^2(\gamma_j)\\ &\le e^{-\epsilon k}\prod_{\gamma\in\Gamma_{\mathrm{loop}}}\big(1+e^{\epsilon|\gamma|}\mathbb{E}w^2(\gamma)\big)\\ &\le \beta^{2k}\exp\Big(-k\log(e^{\epsilon}\beta^2)+\sum_{j=3}^{\infty}\frac{e^{\epsilon j}\beta^{2j}}{2j}\Big) \le \exp\big(8\sqrt{k}-k\log\beta^{-2}\big).\end{aligned}$$

Since $k > \max\big(\frac{1}{2}(1-\beta^2)^{-2}, (16)^2/(\log\beta^{-2})^2\big)$, we have that $k\log\beta^{-1} \ge 8\sqrt{k}$ and thus

$$\mathbb{E}\Big(\sum_{\gamma\in\Gamma_{\mathrm{sc}}:|\gamma|\ge k} w(\gamma)\Big)^2 \le \exp\big(-k\log\beta^{-1}\big).$$

Finally, setting $C = \max(1, C')$ with $C' > 0$ from the first step concludes the first bound of the lemma. For the second bound, we use the rough estimate

$$\sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:\\ |\gamma|=l,\{i,j\}\in\gamma}} \mathbb{E}\,\frac{w^2(\gamma)}{\tanh^2(\beta g_{ij})} \le N^{-1}\beta^{2l} \le \beta^{2l}$$

and proceed similarly as in the first step. $\square$

As indicated in the introduction, the proof of Theorem 1.1 relies crucially on the approximation (1.8). To make this more precise, we need some preparation. We define the map $\Phi : \{(\gamma,\tau) \in \Gamma_{\text{loop}} \times \Gamma_{\text{sc}} : \gamma \cap \tau = \emptyset\} \to \Gamma_{\text{sc}}$ by

$$\big\{(\gamma,\tau) \in \Gamma_{\text{loop}} \times \Gamma_{\text{sc}} : \gamma \cap \tau = \emptyset\big\} \ni (\gamma,\tau) \mapsto \Phi(\gamma,\tau) := \gamma \circ \tau \in \Gamma_{\text{sc}}. \tag{2.5}$$

Note that $\Phi(\gamma,\tau) \in \Gamma_{\text{sc}}$ due to $\gamma \cap \tau = \emptyset$. Projections onto the first and second coordinates in $\Gamma_{\text{s}} \times \Gamma_{\text{s}}$ are denoted by π_1 and π_2, respectively. Furthermore, for $i_1 \neq i_2$, we set

$$S_{i_1 i_2} := \big\{(\gamma,\tau) \in \Gamma_{\text{loop}} \times \Gamma_{\text{sc}} : \{i_1,i_2\} \in \gamma, |\mathcal{V}_\gamma \cap \mathcal{V}_\tau| \leq 1\big\} \subset \Gamma_{\text{loop}} \times \Gamma_{\text{sc}}. \tag{2.6}$$

Observe that $\gamma \cap \tau = \emptyset$ for $(\gamma,\tau) \in S_{i_1,i_2}$, due to the vertex set constraint $|\mathcal{V}_\gamma \cap \mathcal{V}_\tau| \leq 1$.

Lemma 2.3. *Let Φ be defined as in* (2.5) *and let $S_{i_1 i_2}$ be defined as in* (2.6)*, for $i_1 \neq i_2$. Then $\Phi_{i_1 i_2} := \Phi_{|S_{i_1 i_2}} : S_{i_1 i_2} \to \Gamma_{\text{sc}}$ is injective.*

Proof. We prove the more general property that if $\gamma = \Phi(\gamma_1,\gamma_2) = \gamma_1 \circ \gamma_2$ for $(\gamma_1,\gamma_2) \in S_{i_1 i_2}$ and if there exist $\gamma_1' \in \Gamma_{\text{loop}}$ with $\{i_1,i_2\} \in \gamma_1'$ as well as $\gamma_2' \in \Gamma_{\text{sc}}$ with $\gamma_1' \cap \gamma_2' = \emptyset$ so that γ can also be represented as

$$\gamma = \gamma_1 \circ \gamma_2 = \gamma_1' \circ \gamma_2',$$

then it already follows that $\gamma_1' = \gamma_1$ and, as a consequence, that $\gamma_2' = \gamma_2$ (because $\gamma_1 \cap \gamma_2 = \emptyset$ and $\gamma_1' \cap \gamma_2' = \emptyset$). To show that $\gamma_1' = \gamma_1$, assume w.l.o.g. that

$$\mathcal{V}_{\gamma_1} = \{i_1, i_2, \ldots, i_{k_1}, i_{k_1+1}, \ldots, i_{k_2}\}, \quad \mathcal{E}_{\gamma_1} = \big\{\{i_1,i_2\}, \ldots, \{i_{k_1-1}, i_{k_1}\}, \ldots, \{i_{k_2}, i_1\}\big\}$$

for suitable $k_1, k_2 \in \mathbb{N}, k_1 \leq k_2$. Since $\gamma = \Phi_{i_1 i_2}(\gamma_1,\gamma_2)$, $\gamma_1 \in \Gamma_{\text{loop}}$ contains at most one vertex of γ that has degree greater than two, and this happens if and only if γ_1 and γ_2 share exactly one vertex, $|\mathcal{V}_{\gamma_1} \cap \mathcal{V}_{\gamma_2}| = 1$. In this case, let us assume that the vertex is given by $i_{k_1} \in [N]$, i.e. $n_{i_{k_1}}(\gamma) > 2$. In case $|\mathcal{V}_{\gamma_1} \cap \mathcal{V}_{\gamma_2}| = 0$, on the other hand, all the vertices of γ that are contained in γ_1 have degree equal to two.

Consider now $\gamma_1' \in \Gamma_{\text{loop}}$. By assumption, we know $\{i_1,i_2\} \in \mathcal{E}_{\gamma_1} \cap \mathcal{E}_{\gamma_1'}$. We now argue inductively that every other edge of γ_1 is also contained in γ_1'. Note that this implies $\gamma_1 = \gamma_1'$, because $\gamma_1, \gamma_1' \in \Gamma_{\text{loop}}$. So, suppose first by contradiction that $\{i_1, i_{k_2}\} \notin \mathcal{E}_{\gamma_1'}$, then the degree of i_1 must satisfy $n_{i_1}(\gamma) > 2$, because $\{i_1,i_2\}, \{i_1,i_{k_2}\} \in \mathcal{E}_{\gamma_1}$ and because γ_1' contains at least one further edge containing $i_1 \in \mathcal{V}_{\gamma_1} \cap \mathcal{V}_{\gamma_1'}$. From the proceeding paragraph, we conclude $k_1 = 1$ and $|\mathcal{V}_{\gamma_1} \cap \mathcal{V}_{\gamma_2}| = 1$. Now, if $\{i_{k-1}, i_k\} \in \mathcal{E}_{\gamma_1} \cap \mathcal{E}_{\gamma_1'}$ for all $3 \leq k \leq k_2$ remaining edges of γ_1, then we must also have that $n_{i_{k_2}}(\gamma) > 2$, by the same argument. But this means that γ_1 contains two vertices of degree greater than two, which contradicts the fact that $\gamma = \Phi_{i_1 i_2}(\gamma_1,\gamma_2)$. On the other hand, if we find some additional edge $\{i_{k-1}, i_k\} \notin \gamma_1'$ for some $3 \leq k \leq k_2$, we can choose the smallest such k and we conclude $n_{i_{k-1}}(\gamma) > 2$ in addition to $n_{i_1}(\gamma) > 2$, contradicting once again the fact that $\gamma = \Phi_{i_1 i_2}(\gamma_1,\gamma_2)$. We thus conclude that both $\{i_1,i_2\}, \{i_1,i_{k_2}\} \in \mathcal{E}_{\gamma_1} \cap \mathcal{E}_{\gamma_1'}$. But now it is clear that we can repeat the previous argument, assuming in the next step by contradiction that $\{i_{k_2}, i_{k_2-1}\} \notin \mathcal{E}_{\gamma_1'}$. This yields $\{i_1,i_2\}, \{i_1,i_{k_2}\}, \{i_{k_2}, i_{k_2-1}\} \in \mathcal{E}_{\gamma_1} \cap \mathcal{E}_{\gamma_1'}$ and proceeding iteratively this way finally proves that $\mathcal{E}_{\gamma_1} \subset \mathcal{E}_{\gamma_1'}$. $\square$

Lemma 2.3 shows that $\Phi_{ij} : S_{ij} \to \Phi(S_{ij})$ is a bijection and we write

$$\begin{aligned}
&\sum_{\{i,j\}\in\gamma\in\Gamma_{\mathrm{sc}}} \frac{w(\gamma)}{\tanh(\beta g_{ij})} \\
&= \sum_{\substack{\gamma=\gamma_1\circ\gamma_2\in\Phi(S_{ij}):\\ \{i,j\}\in\gamma_1\in\Gamma_{\mathrm{loop}},\gamma_2\in\Gamma_{\mathrm{sc}},\\ \gamma_1\cap\gamma_2=\emptyset,|\mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}|\le 1}} \frac{w(\gamma)}{\tanh(\beta g_{ij})} + \sum_{\{i,j\}\in\gamma\in\Gamma_{\mathrm{sc}}\setminus\Phi(S_{ij}):} \frac{w(\gamma)}{\tanh(\beta g_{ij})} =: \Lambda_{ij} + R_{ij}^{(1)}.
\end{aligned} \tag{2.7}$$

In the following, we analyze Λ_{ij} and $R_{ij}^{(1)}$ separately. We find that the second term on the r.h.s. in (2.7) is typically of order $O(N^{-3/2})$ while Λ_{ij} is close to the r.h.s. in (1.8). To see this latter fact, recalling that $\Phi_{ij} : S_{ij} \to \Phi(S_{ij})$ is a bijection and that $S_{ij} \subset \Gamma_{\mathrm{loop}} \times \Gamma_{\mathrm{sc}}$, we can split the summation in Λ_{ij} as

$$\begin{aligned}
\Lambda_{ij} &= \sum_{\substack{\gamma'=\gamma\circ\tau\in\Phi(S_{ij}):\\ \gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma,\tau\in\Gamma_{\mathrm{sc}},\\ \gamma\cap\tau=\emptyset,|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\le 1}} \frac{w(\gamma)}{\tanh(\beta g_{ij})}\, w(\tau) \\
&= \sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma} \frac{w(\gamma)}{\tanh(\beta g_{ij})} \Bigg(\sum_{\tau\in\Gamma_{\mathrm{sc}}:\gamma\circ\tau\in\Phi(S_{ij})} w(\tau) \Bigg).
\end{aligned}$$

Now, for fixed cycle $\gamma \in \Gamma_{\mathrm{loop}}$ with $\{i,j\} \in \gamma$, we know that $\gamma\circ\tau \in \Phi(S_{ij})$ or, equivalently, $(\gamma,\tau) \in S_{ij}$, if and only if $|\mathcal{V}_\gamma \cap \mathcal{V}_\tau| \le 1$ (implying $\gamma \cap \tau = \emptyset$) so that

$$\begin{aligned}
\sum_{\tau\in\Gamma_{\mathrm{sc}}:\gamma\circ\tau\in\Phi(S_{ij})} w(\tau) &= \sum_{\tau\in\Gamma_{\mathrm{sc}}:(\gamma,\tau)\in S_{ij}} w(\tau) = \sum_{\tau\in\Gamma_{\mathrm{sc}}} w(\tau) - \sum_{\substack{\tau\in\Gamma_{\mathrm{sc}}:\gamma\cap\tau=\emptyset,\\ |\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\ge 2}} w(\tau) - \sum_{\tau\in\Gamma_{\mathrm{sc}}:\gamma\cap\tau\neq\emptyset} w(\tau) \\
&= \sum_{\tau\in\Gamma_{\mathrm{sc}}} w(\tau) - \sum_{\tau\in\Gamma_{\mathrm{sc}}:|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\ge 2} w(\tau).
\end{aligned}$$

Here, we used in the second step that two graphs that share an edge also share at least two vertices. Since the first term on the r.h.s. in the previous equation is $\widehat{Z}_N$, we obtain

$$\Lambda_{ij} = \widehat{Z}_N \sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma} \frac{w(\gamma)}{\tanh(\beta g_{ij})} - R_{ij}^{(2)} - R_{ij}^{(3)} - R_{ij}^{(4)} + R_{ij}^{(5)},$$

where the errors $R_{ij}^{(2)}, R_{ij}^{(3)}, R_{ij}^{(4)}, R_{ij}^{(5)}$ are given by

$$
\begin{aligned}
R_{ij}^{(2)} &:= \sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma} \frac{w(\gamma)\mathbf{1}_{\{|\mathcal{V}_\gamma|<k_N^2\}}}{\tanh(\beta g_{ij})} \sum_{\tau\in\Gamma_{\mathrm{sc}}:|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\geq 2} w(\tau)\mathbf{1}_{\{|\mathcal{V}_\tau|<k_N^4\}}\,, \\
R_{ij}^{(3)} &:= \sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma} \frac{w(\gamma)\mathbf{1}_{\{|\mathcal{V}_\gamma|<k_N^2\}}}{\tanh(\beta g_{ij})} \sum_{\tau\in\Gamma_{\mathrm{sc}}:|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\geq 2} w(\tau)\mathbf{1}_{\{|\mathcal{V}_\tau|\geq k_N^4\}}\,, \\
R_{ij}^{(4)} &:= \sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma} \frac{w(\gamma)\mathbf{1}_{\{|\mathcal{V}_\gamma|\geq k_N^2\}}}{\tanh(\beta g_{ij})} \sum_{\tau\in\Gamma_{\mathrm{sc}}} w(\tau)\,, \\
R_{ij}^{(5)} &:= \sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma} \frac{w(\gamma)\mathbf{1}_{\{|\mathcal{V}_\gamma|\geq k_N^2\}}}{\tanh(\beta g_{ij})} \sum_{\tau\in\Gamma_{\mathrm{sc}}:\gamma\circ\tau\in\Phi(S_{ij})} w(\tau)
\end{aligned}
\tag{2.8}
$$

for fixed k_N, specified below. In addition to $R_{ij}^{(1)}, R_{ij}^{(2)}, R_{ij}^{(3)}, R_{ij}^{(4)}$ and $R_{ij}^{(5)}$, we set

$$
\begin{aligned}
R_{ij}^{(6)} &:= \big(\beta g_{ij} - \tanh(\beta g_{ij})\big) + \sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma} \Bigg(\prod_{e\in\gamma:e\neq\{i,j\}} \beta g_e - \frac{w(\gamma)}{\tanh(\beta g_{ij})}\Bigg), \\
R_{ij}^{(7)} &:= \tanh^2(\beta g_{ij})\Bigg(\tanh(\beta g_{ij}) + \sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma} \frac{w(\gamma)}{\tanh(\beta g_{ij})}\Bigg).
\end{aligned}
\tag{2.9}
$$

In the rest of this section, our goal is to show that the errors $R_{ij}^{(k)}$ are negligible contributions to the Frobenius norm of $\mathbf{M}$. For most of the bounds, it is sufficient to apply Lemma 2.2 to graphs which have a mildly growing size of order $O((\log N)^{1+\epsilon})$ for a suitable, positive $\epsilon > 0$. This yields faster than polynomial (in N) decay for the $L^2(\Omega)$ norm of large graphs while small graphs can be controlled directly using combinatorial arguments. For definiteness, we denote by k_N from now on the large graph threshold

$$k_N := (\log N)^{3/2}. \tag{2.10}$$

Let us start the analysis by showing that the terms $R_{ij}^{(k)}$ for $k \in \{1,3,4,5,6,7\}$ are small. To this end, we need some preparation. For $k\in\mathbb{N}, l\in\mathbb{N}_0$, we define

$$A_{k,l} := \{\gamma\in\Gamma_{\mathrm{sc}} : |\mathcal{V}_\gamma| = k, |\mathcal{E}_\gamma| = k+l\}, \quad A_l := \{\gamma\in\Gamma_{\mathrm{sc}} : |\mathcal{E}_\gamma| - |\mathcal{V}_\gamma| = l\}, \tag{2.11}$$

so that $A_{k,l}=\emptyset$ if $l > \frac{1}{2}k(k-1)$ (the complete graph of k vertices has $\frac{1}{2}k(k-1)$ edges).

Moreover, for subsets $S\subset\Gamma_{\mathrm{s}}$, let us denote in the sequel by $w(S)$ the sum of weights

$$w(S) := \sum_{\gamma\in S} w(\gamma).$$

Lemma 2.4. *Let $\beta<1$, let $m,n\in\mathbb{N}, l\in\mathbb{N}_0$ and let k_N be defined as in (2.10). Then, there exists $C=C_\beta>0$ such that for every $\epsilon>0$, we have that*

$$
\begin{aligned}
\mathbb{E}[w(A_l)^2] &\leq C\max\big(N^{-l+\epsilon}, e^{-k_N^n\log\beta^{-1}}e^{-l\log\beta^{-1}}\big)\,, \\
\mathbb{E}[w(\{\gamma\in A_l : i_1,i_2,\dots,i_m\in\mathcal{V}_\gamma\})^2] &\leq C\max\big(N^{-l-m+\epsilon}, e^{-k_N^n\log\beta^{-1}}e^{-l\log\beta^{-1}}\big)
\end{aligned}
\tag{2.12}
$$

for all $l \le k_N^n$, N large enough and $i_1, i_2, \dots, i_m \in [N]$. Similarly, we have that

$$\mathbb{E}\bigg[\frac{w(\{\gamma \in A_l : \{i,j\} \in \gamma\})^2}{\tanh^2(\beta g_{ij})}\bigg] \le CN^{-1} \max\big(N^{-l+\epsilon}, e^{-k_N^n \log\beta^{-1}} e^{-l\log\beta^{-1}}\big) \tag{2.13}$$

for all $i, j \in [N]$, $i \neq j$ and N large enough.

Finally, if the number of vertices is bounded by k_N^n, we have the improved bounds

$$\begin{aligned} \mathbb{E}[w(\{\gamma \in A_l : |\mathcal{V}_\gamma| \le k_N^n\})^2]] &\le CN^{-l+\epsilon}, \\ \mathbb{E}[w(\{\gamma \in A_l : |\mathcal{V}_\gamma| \le k_N^n; i_1, i_2, \dots, i_m \in \mathcal{V}_\gamma\})^2] &\le CN^{-l-m+\epsilon} \end{aligned} \tag{2.14}$$

for every $\epsilon > 0$ and N large enough.

Proof. From the $L^2(\Omega)$ orthogonality of different graphs in Γ_{sc}, we get

$$\mathbb{E}[w(A_l)^2] = \sum_{k=0}^{k_N^n} \mathbb{E}[w(A_{k,l})^2] + \mathbb{E}\bigg(\sum_{\gamma\in\Gamma_{\mathrm{sc}}:|\gamma|>k_N+l} w(\gamma)\bigg)^2.$$

Now, we first claim that the number of unlabeled simple closed graphs with $k+l$ edges and k vertices is at most $C(k+l)^{2l}e^{C\sqrt{k+l}}$, for some universal constant $C > 0$ independent of k and l. Assuming this for the moment and combining it with the fact that we can assign labels to an unlabeled graph of k vertices in less than N^k ways, we obtain with

$$\mathbb{E}[w(\gamma)^2] \le (\beta^2/N)^{k+l}$$

for every $\gamma \in A_{k,l}$ that

$$\mathbb{E}[w(A_{k,l})^2] \le C(k+l)^{2l}e^{C\sqrt{k+l}}\, N^k\, (\beta^2/N)^{k+l} \le C(k+l)^{Cl}e^{C\sqrt{k+l}-(k+l)\log\beta^{-2}}N^{-l}.$$

Using this bound and Lemma 2.2, we get for every $\epsilon > 0$

$$\begin{aligned} \mathbb{E}[w(A_l)^2] &\le C(k_N+l)^{Cl}N^{-l}\sum_{k=0}^{k_N^n} e^{C\sqrt{k+l}-(k+l)\log\beta^{-2}} + Ce^{-k_N^n\log\beta^{-1}}e^{-l\log\beta^{-1}} \\ &\le C(k_N+l)^{Cl}N^{-l} + Ce^{-k_N^n\log\beta^{-1}}e^{-l\log\beta^{-1}} \\ &\le N^{-l+\epsilon} + Ce^{-k_N^n\log\beta^{-1}}e^{-l\log\beta^{-1}}. \end{aligned}$$

Here, we used the assumption that $l \le k_N^n$ so that $\log(k_N+l) \ll \log N$ for large $N \in \mathbb{N}$.

The bound (2.12) follows in the same way if we use that for $k \ge m$

$$\begin{aligned} \frac{\mathbb{E}[w(\{\gamma \in A_{k,l} : i_1, \dots, i_m \in \mathcal{V}_\gamma\})^2]}{\mathbb{E}[w(A_{k,l})^2]} &= \frac{|\{V \subset [N] : |V| = k, \{i_1, \dots, i_m\} \subset V\}|}{|\{V \subset [N] : |V| = k\}|} \\ &= \frac{(N-m)(N-m-1)\dots(N-k+1)}{N(N-1)\dots(N-k+1)}. \end{aligned}$$

This follows from $\mathbb{E}[w(\gamma)^2] = \mathbb{E}[w(\gamma')^2]$ for all $\gamma, \gamma' \in A_{k,l}$ and the $L^2(\Omega)$ orthogonality of different graphs in Γ_{sc}. The bound (2.13) follows with the same arguments, noting that we lose a factor of N in the rate as we divide $w(\gamma)$ by $\tanh(\beta g_{ij})$ and, finally, for (2.14), we proceed as above, but we do not need a large graph cutoff. That is, we find

$$\mathbb{E}[w(\{\gamma \in A_l : |\mathcal{V}_\gamma| \le k_N^n\})^2] \le C(k_N + l)^{Cl} N^{-l} \sum_{k=0}^{k_N^n} e^{C\sqrt{k+l}-(k+l)\log\beta^{-2}} \le CN^{-l+\epsilon},$$

uniformly in $l \ge 0$, using that $A_{k,l} = \emptyset$ whenever $l > \frac{1}{2}k_N^n(k_N^n - 1)$ and $k_N^n \ll N$. The second bound in (2.14) follows with the same arguments as the second bound in (2.12).

To conclude the lemma, let us prove the claim that the number of unlabeled simple closed graphs with $k+l$ edges and k vertices is bounded by $C(k+l)^{2l}e^{C\sqrt{k+l}}$. The number of unlabeled simple closed graphs with $k+l$ edges and $k+l$ vertices is bounded from above by the number of solutions $(x_1, x_2, \dots, x_{k+l}) \in \mathbb{N}_0^{k+l}$ of the equation

$$x_1 + 2x_2 + ... + (k+l)x_{k+l} = k + l, \tag{2.15}$$

which corresponds to the number of unlabeled simple closed graphs with $k+l$ edges and $k+l$ vertices allowing in addition for self-loops. Here, $x_i \in \mathbb{N}_0$ denotes the number of loops of size $i \in \mathbb{N}$ contained in the graph. The upper bound (2.15) simply follows from decomposing a given graph into an edge disjoint union of cycles (recall (2.4)). Now, the number of solutions to (2.15) corresponds to the number of partitions of $k+l$, which is bounded by $Ce^{C\sqrt{k+l}}$ (see, for instance, [3, Eq. (5.1.2)]) for some universal constant $C > 0$. To conclude the claim, notice that every unlabeled graph with $k+l$ edges and k vertices can be obtained from a graph of $k+l$ edges and $k+l$ vertices by merging l pairs of vertices which can be done in less than $\binom{l+k}{2l}\frac{(2l)!}{2^l l!} \le (k+l)^{2l}$ ways: this is the number of ways to choose $2l$ from $k+l$ vertices times the number of ways to merge l pairs. $\square$

Lemma 2.5. *Let $\beta < 1$ and let $R_{ij}^{(k)}$, for $k \in \{1, 3, 4, 5, 6, 7\}$ be defined as in* (2.7), (2.8) *and* (2.9)*, respectively. Then, we have for every $\epsilon > 0$ that*

$$\max_{i,j\in[N], i\neq j} \big\| R_{ij}^{(k)} \big\|_{L^2(\Omega)} + \max_{i,j\in[N], i\neq j} \big\| \widehat{Z}_N^{-1} R_{ij}^{(4)} \big\|_{L^2(\Omega)} \le N^{-3/2+\epsilon}$$

for all $k \in \{1, 5, 6, 7\}$ and sufficiently large N . Moreover, we have that

$$\lim_{N\to\infty} \mathbb{P}\Big(\max_{i,j\in[N], i\neq j} \big|R_{ij}^{(3)}\big| > N^{-\log N} \Big) = 0.$$

Proof. We prove the bounds for fixed $i, j \in [N]$ with $i \neq j$ and, as becomes clear from the bounds below, all constants are independent of i, j, implying the lemma.

Let us start with the $L^2(\Omega)$ bounds and consider first $R_{ij}^{(1)}$, defined in (2.7). Recalling the definitions in (2.11), we use the $L^2(\Omega)$ orthogonality of different graphs and bound

$$\mathbb{E}\big(R_{ij}^{(1)}\big)^2 \le \sum_{k=3}^{k_N} \sum_{l=2}^{\frac{1}{2}k_N(k_N-1)} \sum_{\gamma \in A_{k,l} : \{i,j\}\in\gamma} \mathbb{E}\, \frac{w(\gamma)^2}{\tanh^2(\beta g_{ij})} + Ce^{-k_N \log\beta^{-1}}, \tag{2.16}$$

where we applied Lemma 2.2 for the large graph contributions. Moreover, we used the fact that for $\gamma \in \Gamma_{\text{sc}} \setminus \Phi(S_{ij})$ with $\{i,j\} \in \gamma$, we find a decomposition $\gamma = \gamma' \circ \tau$ such that $\{i,j\} \in \gamma' \in \Gamma_{\text{loop}}, \tau \in \Gamma_{\text{sc}}, \gamma' \cap \tau = \emptyset$ and $|\mathcal{V}_{\gamma'} \cap \mathcal{V}_\tau| \geq 2$. In particular, we have that $\mathcal{E}_\gamma = \frac{1}{2}\sum_{i=1}^N n_i(\gamma) \geq \mathcal{V}_\gamma + 2$, because at least two vertices have degree greater or equal to four, justifying the upper bound in (2.16). Now, applying Lemma 2.4, we conclude

$$\begin{aligned}\mathbb{E}\big(R_{ij}^{(1)}\big)^2 &\leq CN^{-3+\delta} \sum_{k=3}^{k_N} \sum_{l=2}^{k_N^2} \max\big(N^{-(l-2)}, e^{-k_N \log \beta^{-1/2}} e^{-l \log \beta^{-1}}\big) + Ce^{-k_N \log \beta^{-1}} \\ &\leq CN^{-3+\delta} k_N + Ce^{-k_N \log \beta^{-1}} \leq CN^{-3+\epsilon}\end{aligned}$$

for every $\epsilon > \delta > 0$ and for $N \in \mathbb{N}$ large enough. The same argument implies that

$$\big\| R_{ij}^{(5)} \big\|_{L^2(\Omega)} \leq N^{-3/2+\epsilon},$$

recalling that $\gamma \circ \tau \in \Gamma_{\text{sc}}$ if $\gamma \in \Gamma_{\text{loop}}$ and $\tau \in \Gamma_{\text{sc}}$ are such that $\gamma \circ \tau \in \Phi(S_{ij})$.

The $L^2(\Omega)$ estimates on $R_{ij}^{(6)}$ and $R_{ij}^{(7)}$ are straightforward, combining the second moment computation with the Taylor expansion

$$\tanh(\beta g_{uv}) - \beta g_{uv} = -\int_0^1 ds\, \tanh^2(s\beta g_{uv})\beta g_{uv} = O(g_{uv}^3),$$

and the bound on $\widehat{Z}_N^{-1} R_{ij}^{(4)}$ is obtained from Lemma 2.2, observing that

$$\widehat{Z}_N^{-1} R_{ij}^{(4)} = \sum_{\substack{\gamma \in \Gamma_{\text{loop}}: \\ \{i,j\} \in \gamma, |\mathcal{V}_\gamma| \geq k_N^2}} \frac{w(\gamma)}{\tanh(\beta g_{ij})},$$

so that for every $\epsilon > 0$ and large enough $N \in \mathbb{N}$

$$\big\| \widehat{Z}_N^{-1} R_{ij}^{(4)} \big\|_{L^2(\Omega)}^2 \leq \sum_{n \geq k_N^2} \sum_{\substack{\gamma \in \Gamma_{\text{loop}}: \\ \{i,j\} \in \gamma, |\gamma| = n}} \mathbb{E}\, \frac{w^2(\gamma)}{\tanh^2(\beta g_{ij})} \leq CN^{-2} \sum_{n \geq k_N^2} (\beta^2)^n \leq CN^{-3+\epsilon}$$

Finally, consider the term $R_{ij}^{(3)}$. Applying Lemma 2.2 and Markov's inequality for fixed $\gamma \in \Gamma_{\text{loop}}$ with $\{i,j\} \in \gamma$, we find

$$\mathbb{P}\bigg(\Big| \sum_{\tau \in \Gamma_{\text{sc}}: |\mathcal{V}_\gamma \cap \mathcal{V}_\tau| \geq 2} w(\tau) \mathbf{1}_{\{|\mathcal{V}_\tau| \geq k_N^4\}} \Big| > N^{-k_N^3} \bigg) \leq N^{-k_N^3}.$$

Since the number of $\gamma \in \Gamma_{\text{loop}}$ such that $|\mathcal{V}_\gamma| < k_N^2$ is bounded by $N^{k_N^2}$ and since

$$\lim_{N\to\infty} \mathbb{P}\big(\max_{\gamma \in \Gamma_{\text{loop}}} |w(\gamma)| \geq N^{-3/2+\varepsilon} \big) = \lim_{N\to\infty} \mathbb{P}\big(\min_{i,j \in [N]: i \neq j} |g_{ij}| \leq N^{-1} \big) = 0$$

for each $\epsilon > 0$ (see [2, Eq. (3.5)] for the first statement; the second follows from standard bounds on Gaussian integrals), a simple union bound implies that

$$\begin{aligned}\max_{i,j\in[N]:i\neq j}|R_{ij}^{(3)}| &= \max_{i,j\in[N]:i\neq j}\Bigg|\sum_{\gamma\in\Gamma_{\text{loop}}:\{i,j\}\in\gamma}\frac{w(\gamma)\mathbf{1}_{\{|\mathcal{V}_\gamma|<k_N^2\}}}{\tanh(\beta g_{ij})}\sum_{\tau\in\Gamma_{\text{sc}}:|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\geq 2}w(\tau)\mathbf{1}_{\{|\mathcal{V}_\tau|\geq k_N^4\}}\Bigg|\\ &\leq N^{-k_N^3}\max_{i,j\in[N]:i\neq j}\sum_{\gamma\in\Gamma_{\text{loop}}:\{i,j\}\in\gamma}\left|\frac{w(\gamma)\mathbf{1}_{\{|\mathcal{V}_\gamma|<k_N^2\}}}{\tanh(\beta g_{ij})}\right|\leq N^{k_N^2-k_N^3}\leq N^{-\log N}\end{aligned}$$

with probability tending to one in the limit $N\to\infty$. □

In order to control the remaining error $R_{ij}^{(2)}$, we need some additional definitions and results. In view of (2.8), it is now useful to consider multi-graphs $\gamma\circ\tau$ (edges may have multiplicity greater than one) which are built up from cycles $\gamma\in\Gamma_{\text{loop}}$ and simple, closed graphs $\tau\in\Gamma_{\text{sc}}$. Given such a multi-graph $\gamma\circ\tau$, we set

$$w(\gamma\circ\tau) := w(\gamma)w(\tau),$$

which is consistent with (2.3). Recalling that Γ_{s} denotes the set of simple graphs, let us denote by $\Gamma_{\text{p}}\subset\Gamma_{\text{s}}$ the set of simple, self-avoiding paths, that is, the set of connected $\gamma\in\Gamma_{\text{s}}$ in which exactly two vertices $i,j\in[N]$, $i\neq j$, have degree $n_i(\gamma)=n_j(\gamma)=1$ (corresponding to the end points of the path) and the remaining vertices have degree two. Every path $\pi\in\Gamma_{\text{p}}$ with at least two edges can be identified uniquely with a cycle $\gamma\in\Gamma_{\text{loop}}$ by connecting its end points and, conversely, removing an edge from a given cycle $\gamma\in\Gamma_{\text{loop}}$ defines a unique path $\pi\in\Gamma_{\text{p}}$ with at least two edges whose end points are the vertices of the edge that has been removed.

Now, consider a pair $\gamma\in\Gamma_{\text{loop}}$ and $\tau\in\Gamma_{\text{sc}}$. Then, we define $\Psi:\Gamma_{\text{loop}}\times\Gamma_{\text{sc}}\to\Gamma_{\text{s}}^2$ by

$$\Psi(\gamma,\tau) := (\eta_1,\eta_2),$$

where $\eta_1=\eta_1(\gamma\circ\tau)$ and $\eta_2=\eta_2(\gamma\circ\tau)$ are such that $\eta_1\cap\eta_2=\emptyset$ and

$$\gamma\circ\tau = \eta_1\circ\eta_2\circ\eta_2. \tag{2.17}$$

In other words, η_1 consists of those edges that have multiplicity one in $\gamma\circ\tau$, and η_2 consists of the edges with multiplicity two in $\gamma\circ\tau$ (notice that $\gamma,\tau\in\Gamma_{\text{sc}}$, so each edge $\{i,j\}$ has multiplicity $n_{ij}(\gamma\circ\tau)\in\{0,1,2\}$). In particular, $\Psi(\gamma,\tau)$ is well-defined.

Given $\eta\in\Gamma_{\text{sc}}$, we define moreover the set $T(\eta,\{i,j\})\subset\Gamma_{\text{loop}}\times\Gamma_{\text{sc}}$ by

$$\begin{aligned}T(\eta,\{i,j\}) := &\big\{(\gamma,\tau)\in\Gamma_{\text{loop}}\times\Gamma_{\text{sc}}:\{i,j\}\in\gamma,|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\geq 2,|\gamma|<k_N^2\big\}\\ &\cap\big\{(\gamma,\tau)\in\Gamma_{\text{loop}}\times\Gamma_{\text{sc}}:\pi_1(\Psi(\gamma,\tau))=\eta\big\}.\end{aligned} \tag{2.18}$$

Let us recall that π_i denotes the i-th coordinate in $\Gamma_{\text{s}}\times\Gamma_{\text{s}}$, for $i\in\{1,2\}$.

Lemma 2.6. *Let $\gamma\in\Gamma_{\text{loop}},\tau\in\Gamma_{\text{sc}}$ and let $\Psi=(\eta_1(\cdot),\eta_2(\cdot)):\Gamma_{\text{loop}}\times\Gamma_{\text{sc}}\to\Gamma_{\text{s}}^2$ be defined as in* (2.17). *Then the following properties hold true:*

(1) $\eta_1 = \eta_1(\gamma \circ \tau) \in \Gamma_{\mathrm{sc}}$ *is a simple closed graph and* $\eta_2 = \eta_2(\gamma \circ \tau) = \gamma \cap \tau$ *is either equal to* $\eta_2 = \gamma \in \Gamma_{\mathrm{loop}}$ *or it is equal to an edge-disjoint union of paths in* Γ_p *whose end points lie in* $\mathcal{V}_{\eta_1}$*. If* $\eta_2 = \gamma$*, then* $\tau = \eta_1 \circ \gamma$*.*

If $|\mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}| \geq 2$*, then* η_2 *can always be written as an edge-disjoint union of paths in* Γ_p *with end points in* $\mathcal{V}_{\eta_1}$ *(including both cases* $\eta_2 = \gamma$ *or* $\eta_2 \neq \gamma$*) and, on the other hand, if* $|\mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}| \leq 1$*, then we have necessarily* $\eta_2 = \gamma$*.*

(2) If $|\mathcal{V}_\gamma \cap \mathcal{V}_\tau| \geq 2$ *and* $|\mathcal{E}_{\eta_1}| - |\mathcal{V}_{\eta_1}| \leq 1$*, then* $\eta_2 \neq \emptyset$*.*

(3) For every $\eta_1 \in \Gamma_{\mathrm{sc}}$ *and* $\eta_2 \in \Gamma_{\mathrm{s}}$ *with* $\eta_1 \cap \eta_2 = \emptyset$*, we have that*

$$|\Psi^{-1}(\{(\eta_1, \eta_2)\})| \leq (1 + |\eta_1|) e^{2(|\mathcal{E}_{\eta_1}| - |\mathcal{V}_{\eta_1}|)}. \tag{2.19}$$

(4) For every $\eta \in \Gamma_{\mathrm{sc}}$*, edge* $\{i,j\}$ *and* $(\gamma, \tau), (\gamma', \tau') \in \bigcup_{\eta \in \Gamma_{\mathrm{sc}}} T(\eta, \{i,j\})$*, we have that*

$$\begin{aligned} &\mathbb{E}[w(\gamma \circ \tau) w(\gamma' \circ \tau')] \neq 0 \quad \Leftrightarrow \quad \mathbb{E}[(w(\eta_1(\gamma \circ \tau)) w(\eta_1(\gamma' \circ \tau'))] \neq 0 \\ \Leftrightarrow \quad &(\gamma, \tau), (\gamma', \tau') \in T(\eta', \{i,j\}) \text{ for some } \eta' \equiv \eta_1(\gamma \circ \tau) = \eta_1(\gamma' \circ \tau') \in \Gamma_{\mathrm{sc}}. \end{aligned}$$

Proof. To prove (1), note that for any $i \in [N]$, the degree $n_i(\eta_1) = n_i(\gamma) + n_i(\tau) - 2n_i(\eta_2)$ is even so that $\eta_1 \in \Gamma_{\mathrm{sc}}$. Since $\eta_2 = \gamma \cap \tau$, the remaining statements follow from the fact that every vertex $i \in \mathcal{V}_{\eta_2}$ has degree either one or two (with regards to η_2). In fact, since $\eta_2 \subset \gamma$, every vertex $i \in \mathcal{V}_{\eta_2}$ has degree two with respect to $\gamma \in \Gamma_{\mathrm{loop}}$ so either both edges $\{i, j_1\}, \{i, j_2\} \in \gamma$ that contain $i \in \mathcal{V}_{\eta_2}$ are contained in η_2 or only one of them. Notice, moreover, that $n_i(\eta_2) = 1$ implies $i \in \mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}$. Indeed, with the previous notation, we can assume w.l.o.g. that $\{i, j_1\} \in \eta_2$ while $\{i, j_2\} \in \eta_1$, i.e. $i \in \mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}$.

Now, suppose first that η_2 contains a cycle $\gamma' \in \Gamma_{\mathrm{loop}}$. Then $\gamma' \subset \eta_2 \subset \gamma$, so in fact $\gamma' = \eta_2 = \gamma$ and, as a consequence of $\gamma \circ \tau = \eta_1 \circ \eta_2 \circ \eta_2$, we obtain that $\tau = \eta_1 \circ \eta_2$. On the other hand, assume that $\eta_2 \neq \gamma$ (in particular, η_2 contains no cycles). Then, η_2 must contain a vertex $i \in \mathcal{V}_{\eta_2}$ with degree $n_i(\eta_2) = 1$ (otherwise η_2 would be an even graph that contains a cycle, by (2.4)). Consider the path connected component $\psi_i \subset \eta_2$ that contains $i \in \mathcal{V}_{\eta_2}$. Then, we can write ψ_i as a finite union

$$\mu_i = \{i, j_1\} \circ \{j_1, j_2\} \circ \ldots \circ \{j_{k-1}, j_k\}$$

for vertices $j_1, \ldots, j_k \in \mathcal{V}_{\eta_2}$ with $j_l \neq j_{l'}$ and $j_l \neq i$ for all $l \neq l'$. The fact that $j_k \neq i$ follows from the assumption that $n_i(\eta_2) = 1$. Similarly, the fact that $j_l \neq j_{l'}$ for all $l \neq l'$ follows from the assumption that η_2 contains no cycle, and it is also clear that $n_{j_k}(\eta_2) = 1$, otherwise we could extend the component ψ_i by another edge that contains $j_k \in \mathcal{V}_{\eta_2}$. By the previous remarks, this implies that $j_k \in \mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}$. Now, checking whether $j_l \in \mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}$ for each l, starting with $j_1 \in \mathcal{V}_{\eta_2}$, it is clear that we can write ψ_i as an edge disjoint union of paths in Γ_{p} with endpoints in $\mathcal{V}_{\eta_1}$.

The previous arguments show that $\eta_2 = \gamma$ (in which case $\tau = \eta_1 \circ \eta_2$) or η_2 can be written as an edge disjoint union of paths in Γ_{p} with endpoints in $\mathcal{V}_{\eta_1}$. Now, suppose $|\mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}| \geq 2$ and suppose that $\eta_2 = \gamma \in \Gamma_{\mathrm{loop}}$. Then, writing

$$\eta_2 = \{j_1, j_2\} \circ \ldots \circ \{j_{k-1}, j_k\} \circ \{j_k, j_1\},$$

we find at least two vertices $j_{l_1}, j_{l_2} \in \mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}$ so that η_2 equals an edge disjoint union of paths in Γ_{p} with endpoints in $\mathcal{V}_{\eta_1}$. Finally, if $|\mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}| \leq 1$, we claim that every vertex $i \in \mathcal{V}_{\eta_2}$ has degree $n_i(\eta_2) = 2$ so that η_2 contains a cycle which implies $\eta_2 = \gamma$. Indeed, if $|\mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}| = 0$, η_2 cannot contain a vertex $i \in \mathcal{V}_{\eta_2}$ with degree $n_i(\eta_2) = 1$, by the previous remarks. On the other hand, if there is a vertex $i \in \mathcal{V}_{\eta_2}$ with degree $n_i(\eta_2) = 1$, then consider the path connected component ψ_i that contains $i \in \mathcal{V}_{\eta_2}$. Arguing as above, we see that ψ_i is a disjoint union of paths in Γ_{p} with endpoints in $\mathcal{V}_{\eta_1}$ and that at least one additional vertex $j_k \in \mathcal{V}_{\eta_2}$ has degree $n_{j_k}(\eta_2) = 1$, contradicting the assumption $|\mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}| \leq 1$. This concludes the proof of (1).

Let us now switch to the proof of (2). Here, we use the identities

$$|\mathcal{E}_\gamma| + |\mathcal{E}_\tau| = |\mathcal{E}_{\eta_1}| + 2|\mathcal{E}_{\eta_2}|, \qquad |\mathcal{V}_\gamma \cup \mathcal{V}_\tau| = |\mathcal{V}_{\eta_1}| + |\mathcal{V}_{\eta_2}| - |\mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}|,$$

which implies (2), because if we assume $\eta_2 = \emptyset$, then

$$|\mathcal{E}_{\eta_1}| - |\mathcal{V}_{\eta_1}| = |\mathcal{E}_\gamma| + |\mathcal{E}_\tau| - |\mathcal{V}_\gamma \cup \mathcal{V}_\tau| = |\mathcal{E}_\gamma| + |\mathcal{E}_\tau| - |\mathcal{V}_\gamma| - |\mathcal{V}_\tau| + |\mathcal{V}_\gamma \cap \mathcal{V}_\tau| \geq |\mathcal{V}_\gamma \cap \mathcal{V}_\tau|.$$

To prove (3), suppose first $\eta_1 = \emptyset$. Then $|\Psi^{-1}(\{(\eta_1, \eta_2)\})| \neq 0$ if and only if $\eta_2 \in \Gamma_{\mathrm{loop}}$, because if η_2 is not a loop, then $\eta_2 \circ \eta_2$ does not contain a simple loop, while $\gamma \circ \tau$ always contains simple loops, for $\gamma \in \Gamma_{\mathrm{loop}}$ and $\tau \in \Gamma_{\mathrm{sc}}$. If $\eta_2 \in \Gamma_{\mathrm{loop}}$, on the other hand, the identity $\gamma \circ \tau = \eta_2 \circ \eta_2$ for $(\gamma, \tau) \in \Gamma_{\mathrm{loop}} \times \Gamma_{\mathrm{sc}}$ implies $\gamma = \eta_2$ (since $\eta_2 \subset \gamma$) and thus $\tau = \eta_2$. This shows in particular that (2.19) is true if $\eta_1 = \emptyset$.

So, assume that $\eta_1 \neq \emptyset$. Since every $(\gamma, \tau) \in \Psi^{-1}(\{(\eta_1, \eta_2)\}) \subset \Gamma_{\mathrm{loop}} \times \Gamma_{\mathrm{sc}}$ contains a cycle $\gamma \subset \eta_1 \circ \eta_2$, which in fact determines τ through $\gamma \circ \tau = \eta_1 \circ \eta_2 \circ \eta_2$, notice that

$$|\Psi^{-1}(\{(\eta_1, \eta_2)\})| \leq |\{\gamma \in \Gamma_{\mathrm{loop}} : \eta_2 \subset \gamma \subset \eta_1 \circ \eta_2\}|,$$

so it is enough to count the number of cycles contained in $\eta_1 \circ \eta_2$. Without loss of generality, we can assume that $\Psi^{-1}(\{(\eta_1, \eta_2)\}) \neq \emptyset$, which implies that $\eta_1 \in \Gamma_{\mathrm{sc}}$ and that η_2 either consists of disjoint paths in Γ_{p} with endpoints in $\mathcal{V}_{\eta_1}$ or η_2 is equal to some loop $\gamma \in \Gamma_{\mathrm{loop}}$. In the latter case, $|\Psi^{-1}(\{(\eta_1, \eta_2)\})| = 1$, by the proof of (1). So, assume that $\eta_1 \in \Gamma_{\mathrm{sc}}$, that $\eta_2 \notin \Gamma_{\mathrm{loop}}$ (so it also does not contain a cycle) and that η_2 consists of edge disjoint paths in Γ_{p} with endpoints in $\mathcal{V}_{\eta_1}$.

Here, we start from an arbitrary vertex $i_1 \in \mathcal{V}_{\eta_1}$ and pick one of i_1's adjacent vertices in η_1. Then, the adjacent vertex $i_2 \in \mathcal{V}_{\eta_1}$ is either not in $\mathcal{V}_{\eta_2}$ or it is an end point $i_2 \in \mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}$ of a path $\pi \in \Gamma_{\mathrm{p}}$ with $\pi \subset \eta_2$ having end points in $\mathcal{V}_{\eta_1}$. In the first case, we continue our walk, going at the next step to an adjacent vertex $i_3 \in \mathcal{V}_{\eta_1}$ (going backwards is not allowed at each step; in particular $i_3 \neq i_1$). In the second case, we also walk to a new vertex in $\mathcal{V}_{\eta_1}$, but here there are two different options: either $i_2 \in \mathcal{V}_{\eta_1} \cap \mathcal{V}_{\eta_2}$ has an adjacent vertex in $\mathcal{V}_{\eta_1}$ (which is not equal to i_1) or we can follow the path π that starts at i_2 and arrives at $i_3 \in \mathcal{V}_{\eta_1}$ (since $\pi \in \Gamma_{\mathrm{p}}$ is simple, we can only walk along one direction on π and we can not miss any cycles in $\eta_1 \circ \eta_2$ proceeding this way). Altogether, we arrive at $i_3 \in \mathcal{V}_{\eta_1}$ and now we can repeat the procedure. This way, we arrive after finitely many steps at some vertex along the way twice. This means we have detected a cycle in $\eta_1 \circ \eta_2$. Counting all possible directions that we can follow at each step, we have

$|\mathcal{V}_{\eta_1}|$ vertices from which we can choose our starting point, and then we have $n_{i_k}(\eta_1)-1$ many directions we can choose at each subsequent step. Since every loop $\gamma \in \eta_1 \circ \eta_2$ will certainly be detected, we find at most

$$|\eta_1| \prod_{k \in \mathcal{V}_{\eta_1}} (n_{i_k}(\eta_1)-1) \leq |\eta_1| e^{\sum_{v \in \mathcal{V}_{\eta_1}} \log(n_{i_k}(\eta_1)-1)} \leq |\eta_1| e^{2(|\mathcal{E}_{\eta_1}|-|\mathcal{V}_{\eta_1}|)}$$

cycles, where we used $\log(x-1) \leq x-2$ for $x \geq 2$ and $|\mathcal{E}_\gamma| = \frac{1}{2}\sum_{i=1}^N n_i(\gamma)$ for $\gamma \in \Gamma_{\mathrm{sc}}$.

The statement in (4) is a consequence of η_1 being simple and $\eta_1 \cap \eta_2 = \emptyset$. $\square$

Recalling the definition of $R_{ij}^{(2)}$ in (2.8), we next want to analyze the $L^2(\Omega)$ norm of

$$\sum_{\substack{(\gamma,\tau)\in\Gamma_{\mathrm{loop}}\times\Gamma_{\mathrm{sc}}: \\ \{i,j\}\in\gamma, |\mathcal{V}_\gamma \cap \mathcal{V}_\tau| \geq 2}} w(\gamma \circ \tau),$$

allowing $\gamma \circ \tau$ to be a multi-graph with edge multiplicities greater than one. To bound the $L^2(\Omega)$ norm, we first change the sum over pairs (γ,τ) to a sum over pairs (η_1,η_2) according to (2.17). Because different pairs (γ,τ) may be mapped to the same image $(\eta_1,\eta_2) = (\eta_1(\gamma\circ\tau), \eta_2(\gamma\circ\tau))$, we need to handle the corresponding multiplicity, which is upper bounded in Lemma 2.6 (3). With this is under control, we first fix η_1 and compute the second moment of the sum over η_2 conditionally on the edges g_e for $e \in \eta_1$, and then we compute the norm of η_1. All in all, we identify two probability costs: first, the fact that γ and τ share at least two vertices comes with a combinatorial cost. If $\eta_2 = \emptyset$ (i.e. γ and τ do not share an edge), this cost will be controlled below using Lemma 2.4. Second, if $\eta \neq \emptyset$, we gain a probabilistic cost from γ and τ sharing an edge. This latter cost improves further if i or j do not lie in $\mathcal{V}_{\eta_1}$, because this means that $\{i,j\} \in \eta_2$ (so that both γ and τ contain $\{i,j\}$). The next lemma deals with the cases in which $\eta_2 \neq \emptyset$.

Lemma 2.7. *Suppose that $|\mathcal{V}_{\eta_1}| \leq \sqrt{N}/2$. For any $A \subset T(\eta_1, \{i,j\})$, we have that*

$$\mathbb{E}\Bigg[\Bigg(\sum_{(\gamma,\tau)\in A} w(\gamma\circ\tau)\Bigg)^2 \Bigg| g_e : e \in \eta_1\Bigg] \leq C(1+|\eta_1|) e^{2(|\mathcal{E}_{\eta_1}|-|\mathcal{V}_{\eta_1}|)} w^2(\eta_1), \tag{2.20}$$

where $\mathcal{V}_{\eta_1}^{\mathrm{c}} = [N] \setminus \mathcal{V}_{\eta_1}$. If we assume $|\mathcal{E}_{\eta_1}| - |\mathcal{V}_{\eta_1}| \leq 1$ (so that $\eta_2 \neq \emptyset$), we have

$$\mathbb{E}\Bigg[\Bigg(\sum_{(\gamma,\tau)\in A} w(\gamma\circ\tau)\Bigg)^2 \Bigg| g_e : e \in \eta_1\Bigg] \leq C(1+|\eta_1|)(|\mathcal{V}_{\eta_1}|^2+1)^2 N^{-2} w^2(\eta_1) \tag{2.21}$$

and, finally, if we assume $i \notin \mathcal{V}_{\eta_1}$ or $j \notin \mathcal{V}_{\eta_1}$, then we have that

$$\mathbb{E}\Bigg[\Bigg(\sum_{(\gamma,\tau)\in A} w(\gamma\circ\tau)\Bigg)^2 \Bigg| g_e : e \in \eta_1\Bigg] \leq C(1+|\eta_1|) e^{2(|\mathcal{E}_{\eta_1}|-|\mathcal{V}_{\eta_1}|)} (|\mathcal{V}_{\eta_1}|+1)^2 N^{-4} w^2(\eta_1). \tag{2.22}$$

Proof. We start with the upper bound

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{(\gamma,\tau)\in A} w(\gamma\circ\tau)\Big)^2 \,\Big|\, g_e : e\in\mathcal{E}_{\eta_1}\Big] \\
&= \mathbb{E}\Big[\Big(\sum_{\eta_2\in S} |A\cap\Psi^{-1}(\eta_1,\eta_2)|\, w(\eta_1)w^2(\eta_2)\Big)^2 \Big|\, g_e : e\in\mathcal{E}_{\eta_1}\Big] \qquad (2.23)\\
&\le (1+|\eta_1|)e^{2(|\mathcal{E}_{\eta_1}|-|\mathcal{V}_{\eta_1}|)} w^2(\eta_1)\mathbb{E}\Big[\Big(\sum_{\eta_2\in S} w^2(\eta_2)\Big)^2\Big],
\end{aligned}$$

where we defined

$$S := \pi_2\big(\Psi(A)\big) = \big\{\pi_2(\Psi(\gamma,\tau)) : (\gamma,\tau)\in A\subset T(\eta_1,\{i,j\})\big\} \subset \Gamma_{\mathrm{s}}$$

and where we used Lemma 2.6 (3) as well as the fact that for all $(\gamma,\tau)\in A$ we have that $w(\gamma\circ\tau) = w(\eta_1)\, w^2(\eta_2(\gamma\circ\tau))$ with $\eta_2(\gamma\circ\tau)\in S$, by definition 2.17. To estimate the r.h.s. in (2.23) from above, we use that $\mathbb{E}[Z^4] = 3\sigma^4$ for centered Gaussian random variables $Z\sim\mathcal{N}(0,\sigma^2)$, which implies

$$\mathbb{E}[w^2(\eta_2)w^2(\eta_2')] = 3^{|\eta_2\cap\eta_2'|}\mathbb{E}[w^2(\eta_2)]\mathbb{E}[w^2(\eta_2')]\,.$$

Define $\mathcal{V}^\circ_{\eta_2} := \mathcal{V}_{\eta_2}\setminus\mathcal{V}_{\eta_1}$ to denote the interior points of the paths in $\eta_2\cap\Gamma_{\mathrm{p}}$ with endpoints in $\mathcal{V}_{\eta_1}$, and denote by η_2^∂ the set

$$\eta_2^\partial := \big\{e\in\eta_2 : e = \{j_1,j_2\} \text{ for } j_1,j_2\in\mathcal{V}_{\eta_1}\cap\mathcal{V}_{\eta_2}\big\}\subset\eta_2.$$

That is, η_2^∂ corresponds to the edges in η_2 that connect two vertices in $\mathcal{V}_{\eta_1}\cap\mathcal{V}_{\eta_2}$. Then

$$|\eta_2\cap\eta_2'| \le 2|\mathcal{V}^\circ_{\eta_2}\cap\mathcal{V}^\circ_{\eta_2'}| + |\eta_2^\partial\cap(\eta_2')^\partial| \le 2|\mathcal{V}^\circ_{\eta_2}\cap\mathcal{V}^\circ_{\eta_2'}| + |\eta_2^\partial| + |(\eta_2')^\partial|,$$

because given an edge in $\eta_2\cap\eta_2'\in\Gamma_{\mathrm{s}}$, it either connects two vertices in $\mathcal{V}_{\eta_1}\cap\mathcal{V}_{\eta_2}$ (and thus in $\mathcal{V}_{\eta_1}\cap\mathcal{V}_{\eta_2'}$) or it contains at least one vertex in $\mathcal{V}^\circ_{\eta_2}\cap\mathcal{V}^\circ_{\eta_2'}$. Hence

$$\mathbb{E}\Big(\sum_{\eta_2\in S} w^2(\eta_2)\Big)^2 \le \sum_{\eta_2\in S}\sum_{\eta_2'\in S} 3^{|\mathcal{V}^\circ_{\eta_2}\cap\mathcal{V}^\circ_{\eta_2'}|}\mathbb{E}[3^{|\eta_2^\partial|}w^2(\eta_2)]\mathbb{E}[3^{|(\eta_2')^\partial|}w^2(\eta_2')]\,.$$

Now we prove that the major contribution to the r.h.s. of the last bound comes from the case when $|\mathcal{V}^\circ_{\eta_2}\cap\mathcal{V}^\circ_{\eta_2'}| = \emptyset$. To this end, let us observe that the expectation $\mathbb{E}[w^2(\eta_2)]$ for $\eta_2\in S$ is uniquely determined by the edge occupation numbers $n_{ij}(\eta_2)$ for $1\le i<j\le N$. Considering two graphs $\eta_2\sim\widetilde{\eta}_2$ to be equivalent if there exists a bijection $\pi:[N]\to[N]$ such that $\pi(\mathcal{V}_{\eta_1}) = \mathcal{V}_{\eta_1}$ and $n_{ij}(\widetilde{\eta}_2) = n_{\pi(i)\pi(j)}(\eta_2)$ for all $i,j\in[N]$, we can rewrite $S = \cup_\iota S(\iota)$ as the disjoint union over equivalence classes ι of unlabeled graphs. Thus

$$\begin{aligned}
\mathbb{E}\Big(\sum_{\eta_2\in S} w^2(\eta_2)\Big)^2 &\le \sum_{\iota,\iota'}\sum_{\eta_2\in S(\iota),\eta_2'\in S(\iota')} 3^{|\mathcal{V}^\circ_{\eta_2}\cap\mathcal{V}^\circ_{\eta_2'}|}\mathbb{E}[3^{|\eta_2^\partial|}w^2(\eta_2)]\mathbb{E}[3^{|(\eta_2')^\partial|}w^2(\eta_2')] \\
&= \sum_{\iota,\iota'}\mathbb{E}[3^{|\iota^\partial|}w^2(\iota)]\mathbb{E}[3^{|(\iota')^\partial|}w^2(\iota')]\sum_{\eta_2\in S(\iota),\eta_2'\in S(\iota')} 3^{|\mathcal{V}^\circ_{\eta_2}\cap\mathcal{V}^\circ_{\eta_2'}|}.
\end{aligned} \qquad (2.24)$$

Now, consider any two vertex sets $V, V' \subset [N] \setminus V_{\eta_1}$ such that $|V|, |V'| < k_N^2$. This applies in particular to $\mathcal{V}^\circ_{\eta_2}, \mathcal{V}^\circ_{\eta_2'}$ for every $\eta_2 \in S(\iota), \eta_2' \in S(\iota')$. Rewriting

$$\sum_{\eta_2 \in S(\iota), \eta_2' \in S(\iota')} 3^{|\mathcal{V}^\circ_{\eta_2} \cap \mathcal{V}^\circ_{\eta_2'}|} = \sum_{\substack{V, V' \subset [N] \setminus V_{\eta_1}: \\ |V|, |V'| \le k_N^2}} 3^{|V \cap V'|} |\{\eta_2 \in S(\iota) : \mathcal{V}^\circ_{\eta_2} = V\}| |\{\eta_2' \in S(\iota') : \mathcal{V}^\circ_{\eta_2'} = V'\}|,$$

notice that $|\{\eta_2 \in S(\iota) : \mathcal{V}^\circ_{\eta_2} = V_1\}| = |\{\eta_2 \in S(\iota) : \mathcal{V}^\circ_{\eta_2} = V_2\}|$ if $|V_1| = |V_2|$. More precisely, if $\eta_2, \eta_2' \in S(\iota)$, then clearly $|\mathcal{V}^\circ_{\eta_2}| = |\mathcal{V}^\circ_{\eta_2'}|$ s.t. $|\{\eta_2 \in S(\iota) : \mathcal{V}^\circ_{\eta_2} = V\}|$ either vanishes (if $|V| \ne |\mathcal{V}^\circ_{\eta_2}|$ for some $\eta_2 \in S(\iota)$) or it is equal to some positive integer if $|V| = |\mathcal{V}^\circ_{\eta_2}|$ for some $\eta_2 \in S(\iota)$). Given $l \in \mathbb{N}$, let us therefore abbreviate

$$\lambda_{\iota,l} := |\{\eta_2 \in S(\iota) : \mathcal{V}^\circ_{\eta_2} = V\}|$$

for some and hence all $V \subset [N] \setminus \mathcal{V}_{\eta_1}$ with $|V| = l$. Then, for $l = |V|, l' = |V'| \le k_N^2$, we have for large N ($\gg k_N^2$ and $\gg \frac{1}{2}\sqrt{N} \ge |\mathcal{V}_{\eta_1}|$) that

$$\begin{aligned}
\frac{|\{(V, V') : |V| = l, |V'| = l', |V \cap V'| = k\}|}{|\{(V, V') : |V| = l, |V'| = l', |V \cap V'| = 0\}|} &= \frac{\binom{N - |\mathcal{V}_{\eta_1}|}{k} \binom{N - |\mathcal{V}_{\eta_1}| - k}{l - k} \binom{N - |\mathcal{V}_{\eta_1}| - l - k}{l' - k}}{\binom{N - |\mathcal{V}_{\eta_1}|}{l} \binom{N - |\mathcal{V}_{\eta_1}| - l}{l'}} \\
&= \frac{l! l'!}{k!(l-k)!(l'-k)!} \frac{(N - |\mathcal{V}_{\eta_1}| - l - k)!}{(N - |\mathcal{V}_{\eta_1}| - l)!} \\
&\le (2k_N^4)^k N^{-k}
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{\eta_2 \in S(\iota), \eta_2' \in S(\iota')} 3^{|\mathcal{V}^\circ_{\eta_2} \cap \mathcal{V}^\circ_{\eta_2'}|} &\le \sum_{l, l' = 0}^{k_N^2} \sum_{k=0}^{k_N^2} \lambda_{\iota,l} \lambda_{\iota,l'} \sum_{\substack{V, V' \subset [N] : |V \cap V'| = k, \\ |V| = l, |V'| = l'}} 3^k \\
&\le \sum_{l, l' = 0}^{k_N^2} \sum_{k=0}^{k_N^2} \lambda_{\iota,l} \lambda_{\iota,l'} \sum_{\substack{V, V' \subset [N] : |V \cap V'| = 0, \\ |V| = l, |V'| = l'}} (6k_N^4)^k N^{-k} \\
&\le \sum_{l, l' = 0}^{k_N^2} \sum_{\substack{V, V' \subset [N]: \\ |V| = l, |V'| = l'}} \lambda_{\iota,l} \lambda_{\iota,l'} \Big(\sum_{k \ge 0} (6k_N^4)^k N^{-k} \Big) \le C \sum_{\eta_2 \in S(\iota), \eta_2' \in S(\iota')}
\end{aligned}$$

and therefore, by (2.24), that

$$\begin{aligned}
\mathbb{E} \Big(\sum_{\eta_2 \in S} w^2(\eta_2) \Big)^2 &\le C \sum_{\iota, \iota'} \sum_{\eta_2 \in S(\iota), \eta_2' \in S(\iota')} \mathbb{E}[3^{|\eta_2^\partial|} w^2(\eta_2)] \mathbb{E}[3^{|(\eta_2')^\partial|} w^2(\eta_2')] \\
&= C \Big(\sum_{\eta_2 \in S} 3^{|\eta_2^\partial|} \mathbb{E} w^2(\eta_2) \Big)^2 .
\end{aligned}$$

Now, using Lemma 2.6 (2), we know that $\eta_2 \neq \emptyset$ if $|\mathcal{E}_{\eta_1}| - |\mathcal{V}_{\eta_1}| \leq 1$ and combining this with Lemma 2.6 (1), we obtain the upper bound

$$\begin{aligned}
\sum_{\eta_2\in S} 3^{|\eta_2^\partial|}\mathbb{E}w^2(\eta_2) &= \sum_{l\geq 1} \sum_{\substack{\eta_2=\nu_1\circ\ldots\circ\nu_l\in S:\\ \nu_m\in\Gamma_{\mathrm{p}},\nu_m\cap\nu_n=\emptyset\,(\forall\, m\neq n),\\ |\mathcal{V}_{\eta_1}\cap\mathcal{V}_{\eta_2}|\geq 2}} 3^{|\eta_2^\partial|}\mathbb{E}\, w^2(\eta_2) + \sum_{\substack{\eta_2\in S\cap\Gamma_{\mathrm{loop}}:\\ |\mathcal{V}_{\eta_1}\cap\mathcal{V}_{\eta_2}|\leq 1}} 3^{|\eta_2^\partial|}\mathbb{E}\, w^2(\eta_2)\\
&\leq \sum_{l\geq 1} 3^l \bigg(\sum_{u,v\in\mathcal{V}_{\eta_1}} \sum_{\nu\in\mathcal{P}_{u,v}} \mathbb{E}w^2(\nu)\bigg)^l + \sum_{\substack{\eta_2\in\Gamma_{\mathrm{loop}}:\\ \{i,j\}\in\eta_2}} \mathbb{E}\, w^2(\eta_2)\\
&\leq C\sum_{l\geq 1} 3^l |\mathcal{V}_{\eta_1}|^{2l} N^{-l} + CN^{-2} \leq C(|\mathcal{V}_{\eta_1}|^2+1)N^{-1},
\end{aligned}$$

where $\mathcal{P}_{u,v}$ denotes the set of self-avoiding paths from vertex u to v. Notice that we used that if $\eta_2 \in S\cap\Gamma_{\mathrm{loop}}$ with $|\mathcal{V}_{\eta_1}\cap\mathcal{V}_{\eta_2}| \leq 1$, then $\{i,j\}\in\eta_2$ (by definition of S) and $|\eta_2^\partial| = 0$. Similarly, $|\eta_2^\partial| \leq l$ if η_2 equals an edge disjoint union of l paths in Γ_{p} with endpoints in $\mathcal{V}_{\eta_1}$. In case $|\mathcal{E}_{\eta_1}| - |\mathcal{V}_{\eta_1}| \geq 2$, we simply obtain the upper bound

$$\sum_{\eta_2\in S} \mathbb{E}w^2(\eta_2) \leq C$$

and, combining the two last bounds, we conclude (2.20) and (2.21).

It remains to prove the last bound (2.22). If $i \notin \mathcal{V}_{\eta_1}$ or $j\notin\mathcal{V}_{\eta_1}$, then $\{i,j\}\notin\eta_1$ so that $\{i,j\}\in\eta_2$ for all $(\gamma,\tau)\in T(\eta_1,\{i,j\})$. Here, we obtain additional decay. If $i\in\mathcal{V}_{\eta_1}$ and $j\notin\mathcal{V}_{\eta_1}$, we can proceed as above, but use instead

$$\begin{aligned}
\sum_{\eta_2\in S} \mathbb{E}w^2(\eta_2) &\leq \sum_{l\geq 1} \sum_{\substack{\eta_2=\nu_1\circ\ldots\circ\nu_l\in S:\\ \nu_m\in\Gamma_{\mathrm{p}},\nu_m\cap\nu_n=\emptyset\,(\forall\, m\neq n),\\ \{i,j\}\in\eta_2, |\mathcal{V}_{\eta_1}\cap\mathcal{V}_{\eta_2}|\geq 2}} 3^{|\eta_2^\partial|}\mathbb{E}\, w^2(\eta_2) + \sum_{\substack{\eta_2\in S\cap\Gamma_{\mathrm{loop}}:\\ |\mathcal{V}_{\eta_1}\cap\mathcal{V}_{\eta_2}|\leq 1}} 3^{|\eta_2^\partial|}\mathbb{E}\, w^2(\eta_2)\\
&\leq \sum_{l\geq 1} 3^l \bigg(\sum_{u\in\mathcal{V}_{\eta_1}} \sum_{\nu\in\mathcal{P}_{j,u}} \mathbb{E}w^2(\{i,j\}\circ\nu)\bigg)\bigg(\sum_{u,v\in\mathcal{V}_{\eta_1}} \sum_{\nu\in\mathcal{P}_{u,v}} \mathbb{E}w(\nu)^2\bigg)^{l-1} + CN^{-2}\\
&\leq C(|\mathcal{V}_{\eta_1}|+1)N^{-2}.
\end{aligned}$$

For the cases $i\notin\mathcal{V}_{\eta_1}$, $j\in\mathcal{V}_{\eta_1}$ and both $i\notin\mathcal{V}_{\eta_1}, j\notin\mathcal{V}_{\eta_1}$, we proceed analogously. $\square$

Lemma 2.8. *Let $\beta<1$ and let $R_{ij}^{(2)}$ be as in* (2.8)*. Then, we have for every $\epsilon>0$ that*

$$\max_{i,j\in[N],i\neq j} \big\|R_{ij}^{(2)}\big\|_{L^2(\Omega)} \leq N^{-3/2+\epsilon}$$

Proof. Recalling the definition of $R_{ij}^{(2)}$ in (2.8), notice that the weights $w(\gamma\circ\tau)$ are divided by the edge weight $w(\{i,j\}) = \tanh(\beta g_{ij})$. To apply the previous Lemma 2.7 directly, it is helpful to compare $R_{ij}^{(2)}$ first with the corresponding error term in which we do not divide the multi-graph weights by $\tanh(\beta g_{ij})$. To this end, denote by $\mathbb{E}_{ij}(\cdot)$ the expectation with regards to g_{ij}. Then, we see that

$$\begin{aligned}
&\mathbb{E}_{ij}(R_{ij}^{(2)})^2 \\
&= \mathbb{E}_{ij}\tanh^2(\beta g_{ij})\Bigg(\sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma}\frac{w(\gamma)\mathbf{1}_{\{|\mathcal{V}_\gamma|<k_N^2\}}}{\tanh(\beta g_{ij})}\sum_{\substack{\tau\in\Gamma_{\mathrm{sc}}:\\|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\ge2,\{i,j\}\in\tau}} w(\tau\setminus\{i,j\})\mathbf{1}_{\{|\mathcal{V}_\tau|<k_N^4\}}\Bigg)^2 \\
&\quad+\Bigg(\sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma}\frac{w(\gamma)\mathbf{1}_{\{|\mathcal{V}_\gamma|<k_N^2\}}}{\tanh(\beta g_{ij})}\sum_{\substack{\tau\in\Gamma_{\mathrm{sc}}:\\|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\ge2,\{i,j\}\notin\tau}} w(\tau\setminus\{i,j\})\mathbf{1}_{\{|\mathcal{V}_\tau|<k_N^4\}}\Bigg)^2,
\end{aligned}$$

because the weight $\tanh(\beta g_{ij})$ occurs in each summand either with multiplicity zero or one. By comparison, using the non-negativity of the cross terms by Wick's rule, we get

$$\begin{aligned}
&\mathbb{E}_{ij}\tanh^2(\beta g_{ij})(R_{ij}^{(2)})^2 \\
&\ge \mathbb{E}_{ij}\tanh^4(\beta g_{ij})\Bigg(\sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma}\frac{w(\gamma)\mathbf{1}_{\{|\mathcal{V}_\gamma|<k_N^2\}}}{\tanh(\beta g_{ij})}\sum_{\substack{\tau\in\Gamma_{\mathrm{sc}}:\\|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\ge2,\{i,j\}\in\tau}} w(\tau\setminus\{i,j\})\mathbf{1}_{\{|\mathcal{V}_\tau|<k_N^4\}}\Bigg)^2 \\
&\quad+\mathbb{E}_{ij}\tanh^2(\beta g_{ij})\Bigg(\sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma}\frac{w(\gamma)\mathbf{1}_{\{|\mathcal{V}_\gamma|<k_N^2\}}}{\tanh(\beta g_{ij})}\sum_{\substack{\tau\in\Gamma_{\mathrm{sc}}:\\|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\ge2,\{i,j\}\notin\tau}} w(\tau\setminus\{i,j\})\mathbf{1}_{\{|\mathcal{V}_\tau|<k_N^4\}}\Bigg)^2 \\
&\ge \mathbb{E}_{ij}\tanh^2(\beta g_{ij})\Big(\mathbb{E}_{ij}(R_{ij}^{(2)})^2\Big)\ge\frac{1}{2}\beta^2N^{-1}\mathbb{E}_{ij}(R_{ij}^{(2)})^2
\end{aligned}$$

for N large enough, where we used that $\mathbb{E}_{ij}\tanh^4(\beta g_{ij})\ge\big(\mathbb{E}_{ij}\tanh^2(\beta g_{ij})\big)^2$.

Using the above observation and once again the non-negativity of cross-terms by Wick's rule, we obtain the upper bound

$$\mathbb{E}(R_{ij}^{(2)})^2\le CN\,\mathbb{E}\Bigg(\sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma}\ \sum_{\tau\in\Gamma_{\mathrm{sc}}:|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\ge2} w(\gamma\circ\tau)\mathbf{1}_{\{|\mathcal{V}_\gamma|,|\mathcal{V}_\tau|<k_N^4\}}\Bigg)^2.$$

Using the definition (2.18) and applying Lemma 2.6 (4), we get

$$\begin{aligned}
&\mathbb{E}\Bigg(\sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma}\ \sum_{\tau\in\Gamma_{\mathrm{sc}}:|\mathcal{V}_\gamma\cap\mathcal{V}_\tau|\ge2} w(\gamma\circ\tau)\mathbf{1}_{\{|\mathcal{V}_\gamma|,|\mathcal{V}_\tau|<k_N^4\}}\Bigg)^2 \\
&\le \mathbb{E}\Bigg(\sum_{\eta_1\in\Gamma_{\mathrm{sc}}}\ \sum_{(\gamma,\tau)\in T(\eta_1,\{i,j\})} w(\eta_1)\,w^2(\eta_2(\gamma\circ\tau))\mathbf{1}_{\{|\mathcal{V}_\gamma|,|\mathcal{V}_\tau|<k_N^4\}}\Bigg)^2 \\
&\le \sum_{\eta_1\in\Gamma_{\mathrm{sc}}:|\mathcal{V}_{\eta_1}|<2k_N^4}\mathbb{E}\Bigg(\sum_{(\gamma,\tau)\in T(\eta_1,\{i,j\})} w^2(\eta_2(\gamma\circ\tau))\mathbf{1}_{\{|\mathcal{V}_\gamma|,|\mathcal{V}_\tau|<k_N^4\}}\Bigg)^2\,\mathbb{E}\,w^2(\eta_1).
\end{aligned}$$

Now, we apply (2.14) from Lemma 2.4 together with Lemma 2.7 s.t. on the one hand

$$\begin{aligned}
&\sum_{\substack{\eta_1\in\Gamma_{\mathrm{sc}}:\{i,j\}\subset\mathcal{V}_{\eta_1},\\ |\mathcal{V}_{\eta_1}|<2k_N^4}} \mathbb{E}\bigg(\sum_{(\gamma,\tau)\in T(\eta_1,\{i,j\})} w^2(\eta_2(\gamma\circ\tau))\mathbf{1}_{\{|\mathcal{V}_\gamma|,|\mathcal{V}_\tau|<k_N^4\}}\bigg)^2\,\mathbb{E}\,w^2(\eta_1)\\
&\le CN^{-2}k_N^{24}\sum_{0\le l\le 1}\sum_{\substack{\eta_1\in\Gamma_{\mathrm{sc}}:\{i,j\}\subset\mathcal{V}_{\eta_1},\\ |\mathcal{V}_{\eta_1}|<2k_N^4,\eta_1\in A_l}}\mathbb{E}\,w^2(\eta_1)+Ck_N^{24}\sum_{l=2}^{2k_N^8}\sum_{\substack{\eta_1\in\Gamma_{\mathrm{sc}}:\{i,j\}\subset\mathcal{V}_{\eta_1},\\ |\mathcal{V}_{\eta_1}|<2k_N^4,\eta_1\in A_l}} e^{2l}\,\mathbb{E}\,w^2(\eta_1)\\
&\le CN^{-4+\epsilon}k_N^{24}+CN^{-4+\epsilon}k_N^{32}\sum_{l\ge 2}e^{2l}N^{-(l-2)}\le CN^{-4+\epsilon}
\end{aligned}$$

for $\epsilon>0$ small enough. On the other hand, the improved bounds (2.22) imply similarly

$$\sum_{\substack{\eta_1\in\Gamma_{\mathrm{sc}}:\{i,j\}\cap\mathcal{V}^c_{\eta_1}\neq\emptyset,\\ |\mathcal{V}_{\eta_1}|<2k_N^4}} \mathbb{E}\bigg(\sum_{(\gamma,\tau)\in T(\eta_1,\{i,j\})} w^2(\eta_2(\gamma\circ\tau))\mathbf{1}_{\{|\mathcal{V}_\gamma|,|\mathcal{V}_\tau|<k_N^4\}}\bigg)^2\,\mathbb{E}\,w^2(\eta_1)\le CN^{-4+\epsilon},$$

so that, altogether, we arrive at $\mathbb{E}(R_{ij}^{(2)})^2\le CN^{-3+\epsilon}$, as claimed. □

As a corollary of Proposition 2.8, we can reduce the comparison of $\mathbf{M}$ with the resolvent $(1+\beta^2-\beta\mathbf{G})^{-1}$ to that of comparing $\mathbf{P}=(p_{ij})_{1\le i,j\le N}\in\mathbb{R}^{N\times N}$, defined by

$$p_{ij}:=\begin{cases}\beta g_{ij}+\sum_{\gamma\in\Gamma_{\mathrm{loop}}:\{i,j\}\in\gamma}\prod_{e\in\gamma:e\neq\{i,j\}}\beta g_e & : i\neq j\\ 1 & : i=j,\end{cases}\tag{2.25}$$

with $(1+\beta^2-\beta\mathbf{G})^{-1}$. Notice that, for $i\neq j$, we have the graphical representation

$$p_{ij}=\sum_{\gamma\in\Gamma_{\mathrm{p}}:n_i(\gamma)=n_j(\gamma)=1}\ \prod_{e\in\gamma}\beta g_e.$$

The approximation of $\mathbf{M}$ by $\mathbf{P}$ is summarized in the next corollary which gives precise meaning to (1.8) and which proves the first half of Theorem 1.1.

Corollary 2.9. *Let $\beta<1$, let $\boldsymbol{M}\in\mathbb{R}^{N\times N}$ be defined as in* (1.2) *and let $\boldsymbol{P}\in\mathbb{R}^{N\times N}$ be defined as in* (2.25)*. Then, we have in the sense of probability that*

$$\lim_{N\to\infty}\|\boldsymbol{M}-\boldsymbol{P}\|_{\mathrm{F}}=0.$$

Proof. Since $\langle\sigma_i\sigma_i\rangle=1=p_{ii}$, it is enough to compare the off-diagonal elements of $\mathbf{M}$ and $\mathbf{P}$. By the identities (2.2), (2.7), (2.8) and (2.9), we obtain for $i\neq j$ that

$$(\mathbf{M}-\mathbf{P})_{ij}=(1-\tanh^2(\beta g_{ij}))\widehat{Z}_N^{-1}\Big(R_{ij}^{(1)}-R_{ij}^{(2)}-R_{ij}^{(3)}-R_{ij}^{(4)}+R_{ij}^{(5)}\Big)-R_{ij}^{(6)}-R_{ij}^{(7)}.$$

Controlling the operator through the Frobenius norm by Cauchy-Schwarz, and then combining the fact that $(\sum_{k=1}^7 R_{ij}^{(k)})^2 \le C \sum_{k=1}^7 (R_{ij}^{(k)})^2$ with $\tanh^2(.) \le 1$, we obtain

$$\begin{aligned}
&\mathbb{P}\big(\|\mathbf{M}-\mathbf{P}\|_{\mathrm{F}}^2 > \delta^2\big)\\
&\le \mathbb{P}\bigg(C \sum_{1\le i\neq j\le N} \widehat{Z}_N^{-2} \sum_{k=1}^{5} \big(R_{ij}^{(k)}\big)^2 + \sum_{k=6}^{7} \big(R_{ij}^{(k)}\big)^2 > \delta^2\bigg)\\
&\le \sum_{k=1}^{5} \mathbb{P}\bigg(\sum_{1\le i\neq j\le N} \big(R_{ij}^{(k)}\big)^2 > \frac{\widehat{Z}_N^2 \delta^2}{C'}\bigg) + \sum_{k=6}^{7} \mathbb{P}\bigg(\sum_{1\le i\neq j\le N} \big(R_{ij}^{(k)}\big)^2 > \frac{\delta^2}{C'}\bigg).
\end{aligned}$$

Applying Markov's inequality on the r.h.s. of the last bound yields

$$\begin{aligned}
&\mathbb{P}\big(\|\mathbf{M}-\mathbf{P}\|_{\mathrm{op}} > \delta)\big)\\
&\le \mathbb{P}\big(\widehat{Z}_N \le \epsilon\big) + \sum_{k=1}^{5} \mathbb{P}\bigg(\sum_{1\le i\neq j\le N} \big(R_{ij}^{(k)}\big)^2 > \frac{\delta^2\epsilon^2}{C'}\bigg) + C'\delta^{-2} \sum_{1\le i\neq j\le N} \sum_{k=6}^{7} \|R_{ij}^{(k)}\|_{L^2(\Omega)}^2\\
&\le \mathbb{P}\big(\widehat{Z}_N \le \epsilon\big) + C'\delta^{-2} N^2 (1+\epsilon^{-2}\big) \max_{k\in\{1,2,4,5,6,7\}} \max_{i,j\in[N]:i\neq j} \|R_{ij}^{(k)}\|_{L^2(\Omega)}^2\\
&\quad + \mathbb{P}\Big(\max_{i,j\in[N]:i\neq j} |R_{ij}^{(3)}|^2 > N^{-2}\delta^2\epsilon^2/C'\Big)
\end{aligned}$$

for every $\delta, \epsilon > 0$. By Lemmas 2.5 and 2.8, we conclude that

$$\begin{aligned}
\limsup_{N\to\infty} \mathbb{P}\big(\|\mathbf{M}-\mathbf{P}\|_{\mathrm{op}} > \delta\big) &\le \limsup_{N\to\infty} \Big(\mathbb{P}\big(\{\widehat{Z}_N \le \epsilon\}\big) + C'\delta^{-2}(1+\epsilon^{-2}\big)CN^{-1} + o(1)\Big)\\
&= \mathbb{P}\big(Y \le \sigma^2/2 + \log\epsilon\big),
\end{aligned}$$

for every $\delta > 0$. Here $o(1)$ denotes an error with $o(1) \to 0$ as $N \to \infty$ and $Y \sim \mathcal{N}(0, \sigma^2)$ denotes a Gaussian random variable with variance as in Proposition 2.1. Since $\epsilon > 0$ was arbitrary, the claim follows by sending $\epsilon \to 0$. ☐

3 Proof of Theorem 1.1

In this section we conclude the proof of our main result, Theorem 1.1. Throughout this section, we assume that for some $\epsilon > 0$ small enough, we have that $\|\beta\mathbf{G}\|_{\mathrm{op}} \le 2\beta + (1-\beta)^2\epsilon/2$. It is well-known (see e.g. [4]) that this holds true on a set of probability at least $1 - o(1)$, for some error $o(1)$ (that depends on ϵ) such that $\lim_{N\to\infty} o(1) = 0$.

As a consequence of Corollary 2.9, Theorem 1.1 follows if we prove that $\mathbf{P} \in \mathbb{R}^{N\times N}$, defined in (2.25), converges in norm to $(1+\beta^2-\beta\mathbf{G})^{-1}$. The assumption on $\|\mathbf{G}\|_{\mathrm{op}}$ implies $\|(1+\beta^2-\beta\mathbf{G})^{-1}\|_{\mathrm{op}} \le (1-\beta)^{-2}(1+\epsilon)$ s.t. it is enough to prove that the matrix

$$\mathbf{Q} = (q_{ij})_{1\le i,j\le N} := \mathbf{P}(1+\beta^2-\beta\mathbf{G}) - \mathrm{id}_{\mathbb{R}^N} \in \mathbb{R}^{N\times N} \tag{3.1}$$

converges in norm to zero. To prepare the proof, we split $\mathbf{Q}$ into several contributions and treat each of the terms separately. Starting with the diagonal elements, we find

$$\begin{aligned} q_{ii} &= \beta^2\Big(1-\sum_k g_{ik}^2\Big) - \sum_k \sum_{\{i,k\}\in\gamma\in\Gamma_{\text{loop}}} \beta g_{ik} \prod_{e\in\gamma: e\neq\{i,k\}} \beta g_e \\ &= \frac{\beta^2}{N} - \beta^2 \sum_k \big(g_{ik}^2 - \mathbb{E}g_{ik}^2\big) - \sum_k \sum_{\{i,k\}\in\gamma\in\Gamma_{\text{loop}}} \prod_{e\in\gamma} \beta g_e \\ &= \frac{\beta^2}{N} p_{ii} - \beta^2 \sum_k \big(g_{ik}^2 - \mathbb{E}g_{ik}^2\big) p_{ii} - 2 \sum_{\gamma\in\Gamma_{\text{loop}}:\, i\in\mathcal{V}_\gamma} \prod_{e\in\gamma} \beta g_e. \end{aligned} \tag{3.2}$$

for $i\in[N]$. To ease the notation, here and from now on all summations over matrix indices run over $[N]$ (recall also that $g_{ii}\equiv 0$). For $i,j\in[N]$ with $i\neq j$, we compute

$$\begin{aligned} q_{ij} &= \beta^3 g_{ij} - \sum_{k:k\neq i} \sum_{\{i,k\}\in\gamma\in\Gamma_{\text{loop}}} \Big(\prod_{e\in\gamma: e\neq\{i,k\}} \beta g_e \Big) \beta g_{kj} \\ &\quad + (1+\beta^2) \sum_{\{i,j\}\in\gamma\in\Gamma_{\text{loop}}} \prod_{e\in\gamma: e\neq\{i,j\}} \beta g_e - \sum_k \beta g_{ik}\beta g_{kj} \\ &= \beta^3 g_{ij}\Big(1 - \sum_{k:k\neq i} g_{kj}^2\Big) - \sum_{k:k\neq i} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:\\ \{i,k\}\in\gamma, |\gamma|\geq 4}} \Big(\prod_{e\in\gamma: e\neq\{i,k\}} \beta g_e \Big) \beta g_{kj} \\ &\quad + \beta^2 \sum_{\{i,j\}\in\gamma\in\Gamma_{\text{loop}}} \prod_{e\in\gamma: e\neq\{i,j\}} \beta g_e + \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma, |\gamma|\geq 5}} \prod_{e\in\gamma: e\neq\{i,j\}} \beta g_e, \end{aligned}$$

where, in the second equality, we inserted the representations

$$\sum_{k:k\neq i} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:\\ \{i,k\}\in\gamma, |\gamma|=3}} \Big(\prod_{e\in\gamma: e\neq\{i,k\}} \beta g_e \Big) = \sum_{k:k\neq i} \sum_l \beta g_{il}\beta g_{lk}$$

and

$$\sum_{\substack{\gamma\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma, |\gamma|=4}} \prod_{e\in\gamma: e\neq\{i,j\}} \beta g_e = \sum_{k,l: k\neq i, l\neq j} \beta g_{il}\beta g_{lk}\beta g_{kj}.$$

Next, we decompose (keeping in mind that $g_{kj}\equiv 0$ for $k=j$)

$$\begin{aligned} \sum_{k:k\neq i} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:\\ \{i,k\}\in\gamma, |\gamma|\geq 4}} \Big(\prod_{e\in\gamma: e\neq\{i,k\}} \beta g_e \Big) \beta g_{kj} &= \beta^2 \sum_{k:k\neq i} g_{kj}^2 \sum_{\substack{\gamma\in\Gamma_{\text{loop}}: |\gamma|\geq 4,\\ \{i,k\}\in\gamma, \{k,j\}\in\gamma}} \Big(\prod_{e\in\gamma: e\neq\{i,k\},\{k,j\}} \beta g_e \Big) \\ &\quad + \sum_{k:k\neq i} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}: |\gamma|\geq 4,\\ \{i,k\}\in\gamma, \{k,j\}\notin\gamma}} \Big(\prod_{e\in\gamma: e\neq\{i,k\}} \beta g_e \Big) \beta g_{kj}. \end{aligned}$$

Since a cycle $\tau \in \Gamma_{\text{loop}}$ that contains the edges $\{i,j\},\{i,k\},\{k,j\}$ must in fact equal the cycle $\tau = \{i,j\} \circ \{i,k\} \circ \{k,j\}$, which is of length $|\tau| = 3$, we have that

$$\sum_{\substack{\gamma\in\Gamma_{\text{loop}}:|\gamma|\geq 4,\\ \{i,k\}\in\gamma,\{k,j\}\in\gamma}} \Big(\prod_{e\in\gamma:e\neq\{i,k\},\{k,j\}} \beta g_e\Big) = \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:|\gamma|\geq 4\\ \{i,k\},\{k,j\}\in\gamma,\{i,j\}\notin\gamma}} \Big(\prod_{e\in\gamma:e\neq\{i,k\},\{k,j\}} \beta g_e\Big).$$

Now, observe that every cycle $\gamma \in \Gamma_{\text{loop}}$ that contains $\{i,k\},\{k,j\} \in \gamma$ and that does not contain $\{i,j\} \notin \gamma$, can be identified uniquely with a cycle γ' of degree $|\gamma'| = |\gamma| - 1$ with $\{i,j\} \in \gamma'$ and $k \notin \mathcal{V}_{\gamma'}$. In fact, given a cycle γ with these properties, the edges $\mathcal{E}_\gamma \setminus \{\{i,k\},\{k,j\}\}$ determine a unique, self-avoiding path from vertex i to vertex j that avoids vertex k, and closing this path through the edge $\{i,j\}$ yields γ'. Thus

$$\begin{aligned}\sum_{\substack{\gamma\in\Gamma_{\text{loop}}:|\gamma|\geq 4,\\ \{i,k\}\in\gamma,\{k,j\}\in\gamma,}} \Big(\prod_{e\in\gamma:e\neq\{i,k\},\{k,j\}} \beta g_e\Big) &= \sum_{\{i,j\}\in\gamma'\in\Gamma_{\text{loop}}} \Big(\prod_{e\in\gamma':e\neq\{i,j\}} \beta g_e\Big)\\ &\quad - \sum_{\substack{\gamma'\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma',\, k\in\mathcal{V}_{\gamma'}}} \Big(\prod_{e\in\gamma':e\neq\{i,j\}} \beta g_e\Big).\end{aligned}$$

Similarly, suppose that $\gamma \in \Gamma_{\text{loop}}$ with $|\gamma| \geq 4$ is a cycle that contains the edge $\{i,k\} \in \gamma$, but such that $j \notin \mathcal{V}_\gamma$ (in particular$\{k,j\} \notin \gamma$). Such a loop can be identified uniquely with a self-avoiding path from vertex i to vertex k, avoiding vertex j. Such a path, on the other hand, can be identified uniquely with a cycle γ' that contains the edges $\{i,j\}$ and $\{j,k\}$, with $|\gamma'| = |\gamma| + 1$. What this implies is that

$$\begin{aligned}&\sum_{k:k\neq i} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:|\gamma|\geq 4,\\ \{i,k\}\in\gamma,\{k,j\}\notin\gamma}} \Big(\prod_{e\in\gamma:e\neq\{i,k\}} \beta g_e\Big)\beta g_{kj}\\ &= \sum_{k:k\neq i} \sum_{\substack{\gamma'\in\Gamma_{\text{loop}}:|\gamma'|\geq 5\\ \{i,j\},\{j,k\}\in\gamma'}} \prod_{e\in\gamma:e\neq\{i,j\}} \beta g_e + \sum_{k:k\neq i} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:|\gamma|\geq 4,\\ \{i,k\}\in\gamma,\{k,j\}\notin\gamma,j\in\mathcal{V}_\gamma}} \Big(\prod_{e\in\gamma:e\neq\{i,k\}} \beta g_e\Big)\beta g_{kj}\\ &= \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:\\ |\gamma'|\geq 5,\{i,j\}\in\gamma'}} \prod_{e\in\gamma:e\neq\{i,j\}} \beta g_e + \sum_{k:k\neq i} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:|\gamma|\geq 4,\\ \{i,k\}\in\gamma,\{k,j\}\notin\gamma,j\in\mathcal{V}_\gamma}} \Big(\prod_{e\in\gamma:e\neq\{i,k\}} \beta g_e\Big)\beta g_{kj},\end{aligned}$$

where in the second step, we used that the set of cycles that contain $\{i,j\},\{j,k\}$ is disjoint from the set of cycles that $\{i,j\},\{j,k'\}$ for $k \neq k'$. Collecting the previous identities, we conclude for $i \neq j$ that

$$\begin{aligned}
q_{ij} &= \beta^2\Big(1 - \sum_{k:k\neq i} g_{kj}^2\Big)\Big(\beta g_{ij} + \sum_{\{i,j\}\in\gamma\in\Gamma_{\text{loop}}} \prod_{e\in\gamma: e\neq\{i,j\}} \beta g_e\Big)\\
&\quad + \beta^2 \sum_{k:k\neq i} g_{kj}^2 \sum_{\substack{\gamma'\in\Gamma_{\text{loop}}:\\ \{i,j\}\in,\, k\in\mathcal{V}_{\gamma'}}} \prod_{e\in\gamma': e\neq\{i,j\}} \beta g_e\\
&\quad - \sum_{k:k\neq i} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:|\gamma|\geq 4,\\ \{i,k\}\in\gamma,\{k,j\}\notin\gamma, j\in\mathcal{V}_\gamma}} \Big(\prod_{e\in\gamma: e\neq\{i,k\}} \beta g_e\Big)\beta g_{kj} \\
&= \frac{\beta^2}{N} p_{ij} - \beta^2\Big(\sum_k \big(g_{kj}^2 - \mathbb{E} g_{kj}^2\big)\Big)p_{ij} + \beta^2 \sum_{k:k\neq i} g_{kj}^2 \sum_{\substack{\gamma'\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma',\, k\in\mathcal{V}_{\gamma'}}} \prod_{e\in\gamma': e\neq\{i,j\}} \beta g_e\\
&\quad + \beta^2 g_{ij}^2 p_{ij} - \sum_{k:k\neq i} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:|\gamma|\geq 4,\\ \{i,k\}\in\gamma,\{k,j\}\notin\gamma, j\in\mathcal{V}_\gamma}} \Big(\prod_{e\in\gamma: e\neq\{i,k\}} \beta g_e\Big)\beta g_{kj},
\end{aligned} \tag{3.3}$$

where we recall p_{ij} from (2.25). For later reference, let us finally observe that

$$\begin{aligned}
&\sum_{k:k\neq i} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:|\gamma|\geq 4,\\ \{i,k\}\in\gamma,\{k,j\}\notin\gamma, j\in\mathcal{V}_\gamma}} \Big(\prod_{e\in\gamma: e\neq\{i,k\}} \beta g_e\Big)\beta g_{kj}\\
&= \sum_{k:k\neq i,j} \sum_{\substack{(\gamma_1,\gamma_2)\in\Gamma_{\text{loop}}^2:\\ \mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}=\{j\},\{i,j\}\in\gamma_1,\{j,k\}\in\gamma_2}} \Big(\prod_{e\in\gamma_1: e\neq\{i,j\}} \beta g_e\Big)\Big(\prod_{e'\in\gamma_2} \beta g_{e'}\Big)\\
&= \sum_{\substack{\gamma_1\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma_1}} \Big(\prod_{e\in\gamma_1: e\neq\{i,j\}} \beta g_e\Big) \sum_{k:k\neq i,j} \sum_{\substack{\gamma_2\in\Gamma_{\text{loop}}: \{j,k\}\in\gamma_2,\\ \mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}=\{j\}}} \prod_{e'\in\gamma_2} \beta g_{e'}\\
&= 2 \sum_{\substack{\gamma_1\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma_1}} \Big(\prod_{e\in\gamma_1: e\neq\{i,j\}} \beta g_e\Big) \sum_{\substack{\gamma_2\in\Gamma_{\text{loop}}:\\ \mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}=\{j\}}} \prod_{e'\in\gamma_2} \beta g_{e'}.
\end{aligned} \tag{3.4}$$

Indeed, a loop $\gamma \in \Gamma_{\text{loop}}$ that contains the edge $\{i,k\} \in \gamma$ and the vertex $j \in \mathcal{V}_\gamma$, but not the edge $\{j,k\} \notin \gamma$ can be uniquely identified with a pair of self-avoiding walks, one going from vertex i to j avoiding vertex k, and the other going from vertex j to vertex k having length at least two (because $\{j,k\} \notin \gamma$). In particular, the intersection of the two walks is given by vertex j only. Since a self-avoiding walk from vertex i to j can be identified uniquely with a cycle γ_1 that contains $\{i,j\} \in \gamma_1$, and since adding the edge $\{j,k\}$ to the second walk turns it into another cycle γ_2, we can identify $\gamma \circ \{j,k\}$ with $\gamma_1 \circ \gamma_2$, where $\{i,j\} \in \gamma_1$, $\{j,k\} \in \gamma_2$ and $\mathcal{V}_{\gamma_1} \cap \mathcal{V}_{\gamma_2} = \{j\}$. Now, varying k over all $k \in [N] \setminus \{i,j\}$, we obtain the identity (3.4).

The decompositions (3.2) and (3.3) imply the identity

$$\mathbf{Q} = \frac{\beta^2}{N}\,\mathbf{P} + \mathbf{Q}^{(1)} + \mathbf{Q}^{(2)} + \mathbf{Q}^{(3)} + \mathbf{P}\mathbf{Q}^{(4)} + \mathbf{Q}^{(5)}, \tag{3.5}$$

where $\mathbf{P}$ is as in (2.25) and where the $\mathbf{Q}^{(k)} = (q_{ij}^{(k)})_{1\leq i,j\leq N} \in \mathbb{R}^{N\times N}$ are defined by

$$\begin{aligned}
q_{ij}^{(1)} &:= \begin{cases} -2\sum_{\substack{\gamma_1\in\Gamma_{\mathrm{loop}}:\\ \{i,j\}\in\gamma_1}} \Big(\prod_{e\in\gamma_1: e\neq\{i,j\}} \beta g_e\Big) \sum_{\substack{\gamma_2\in\Gamma_{\mathrm{loop}}:\\ \mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}=\{j\}}} \prod_{e'\in\gamma_2} \beta g_{e'} & : i\neq j \\ 0 & : i=j, \end{cases}\\
q_{ij}^{(2)} &:= \begin{cases} 0 & : i\neq j \\ -2\sum_{\gamma\in\Gamma_{\mathrm{loop}}: i\in\mathcal{V}_\gamma} \prod_{e\in\gamma} \beta g_e & : i=j. \end{cases}\\
q_{ij}^{(3)} &:= \begin{cases} \beta^2\sum_{k:k\neq i} g_{kj}^2 \sum_{\substack{\gamma'\in\Gamma_{\mathrm{loop}}:\\ \{i,j\}\in\gamma',\, k\in\mathcal{V}_{\gamma'}}} \prod_{e\in\gamma': e\neq\{i,j\}} \beta g_e & : i\neq j \\ 0 & : i=j, \end{cases}\\
q_{ij}^{(4)} &:= \begin{cases} 0 & : i\neq j \\ -\beta^2\sum_k \big(g_{ik}^2 - \mathbb{E} g_{ik}^2\big) & : i=j, \end{cases}\\
q_{ij}^{(5)} &:= \begin{cases} \beta^2 g_{ij}^2 p_{ij} & : i\neq j \\ 0 & : i=j, \end{cases}
\end{aligned} \tag{3.6}$$

Observe that $\mathbf{Q}^{(1)}$ is equal to the matrix with vanishing diagonal and with off-diagonal elements given by the r.h.s. in (3.4). For fixed $k, k_1, k_2 \geq 3$, it is also useful to define $\mathbf{Q}^{(1)}_{>k_1,k_2}$, $\mathbf{Q}^{(1)}_{\leq k_1,k_2}$, $\mathbf{Q}^{(2)}_{>k}$, $\mathbf{Q}^{(2)}_{\leq k}$, $\mathbf{Q}^{(6)}_{>k_1,k_2}$ and $\mathbf{Q}^{(6)}_{\leq k_1,k_2}$ via

$$\begin{aligned}
q^{(1)}_{\leq k_1,k_2,ij} &:= \begin{cases} -2\sum_{\substack{\gamma_1\in\Gamma_{\mathrm{loop}}:\\ \{i,j\}\in\gamma_1, |\gamma_1|\leq k_1}} \big(\prod_{e\in\gamma_1: e\neq\{i,j\}} \beta g_e\big) \sum_{\substack{\gamma_2\in\Gamma_{\mathrm{loop}}: |\gamma_2|\leq k_2,\\ \mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}=\{j\}}} \prod_{e'\in\gamma_2} \beta g_{e'} & : i\neq j \\ 0 & : i=j, \end{cases}\\
q^{(1)}_{>k_1,k_2,ij} &:= \begin{cases} q_{ij}^{(1)} - q^{(1)}_{\leq k_1,k_2,ij} & : i\neq j \\ 0 & : i=j, \end{cases}\\
q^{(2)}_{\leq k,ij} &:= \begin{cases} 0 & : i\neq j \\ -2\sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:\\ |\gamma|\leq k, i\in\mathcal{V}_\gamma}} \prod_{e\in\gamma} \beta g_e & : i=j, \end{cases} \qquad q^{(2)}_{>k,ij} := \begin{cases} 0 & : i\neq j \\ q_{ij}^{(2)} - q^{(2)}_{\leq k,ij} & : i=j, \end{cases}\\
q^{(6)}_{>k_1,k_2,ij} &:= \begin{cases} -2\sum_{\substack{\gamma_1\in\Gamma_{\mathrm{loop}}:\\ \{i,j\}\in\gamma_1, |\gamma_1|>k_1}} \big(\prod_{e\in\gamma_1: e\neq\{i,j\}} \beta g_e\big) \sum_{\substack{\gamma_2\in\Gamma_{\mathrm{loop}}:\\ |\gamma_2|\leq k_2, j\in\mathcal{V}_{\gamma_2}}} \prod_{e'\in\gamma_2} \beta g_{e'} & : i\neq j \\ 0 & : i=j. \end{cases}\\
q^{(6)}_{\leq k_1,k_2,ij} &:= \begin{cases} -2\sum_{\substack{\gamma_1\in\Gamma_{\mathrm{loop}}:\\ \{i,j\}\in\gamma_1, |\gamma_1|\leq k_1}} \big(\prod_{e\in\gamma_1: e\neq\{i,j\}} \beta g_e\big) \sum_{\substack{\gamma_2\in\Gamma_{\mathrm{loop}}: |\gamma_2|\leq k_2,\\ j\in\mathcal{V}_{\gamma_2}, |\mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}|\geq 2}} \prod_{e'\in\gamma_2} \beta g_{e'} & : i\neq j \\ 0 & : i=j, \end{cases}
\end{aligned} \tag{3.7}$$

Notice that, by (3.6) and (3.7), we have the useful decomposition

$$\mathbf{Q}^{(1)} = \mathbf{Q}^{(1)}_{\leq k_1,k_2} + \mathbf{Q}^{(1)}_{>k_1,k_2} = \mathbf{Q}^{(1)}_{>k_1,k_2} + \big(\mathbf{P} - \mathrm{id}_{\mathbb{R}^N} - \beta\mathbf{G}\big)\big(\mathbf{Q}^{(2)} - \mathbf{Q}^{(2)}_{>k_2}\big) - \mathbf{Q}^{(6)}_{>k_1,k_2} - \mathbf{Q}^{(6)}_{\leq k_1,k_2},$$

which implies with (3.5) that for every $k_1, k_2 \geq 3$

$$\begin{aligned} \mathbf{Q} = \mathbf{P}\Big(\frac{\beta^2}{N}\mathrm{id}_{\mathbb{R}^N} + \mathbf{Q}^{(2)} - \mathbf{Q}^{(2)}_{>k_2} + \mathbf{Q}^{(4)}\Big) - \beta\mathbf{G}\mathbf{Q}^{(2)} + \mathbf{Q}^{(3)} + \mathbf{Q}^{(5)} \\ + \mathbf{Q}^{(1)}_{>k_1,k_2} + \big(\mathrm{id}_{\mathbb{R}^N} + \beta\mathbf{G}\big)\mathbf{Q}^{(2)}_{>k_2} - \mathbf{Q}^{(6)}_{>k_1,k_2} - \mathbf{Q}^{(6)}_{\leq k_1,k_2}. \end{aligned} \tag{3.8}$$

The following lemma collects important properties of the various matrix elements.

Lemma 3.1. *Let $\beta < 1$, $\boldsymbol{P}$ as in (2.25), $\boldsymbol{Q}$ as in (3.1) and let $\boldsymbol{Q}^{(k)}$, $k \in \{1, \dots, 5\}$, be defined as in (3.6). Then, there exists a constant $C = C_\beta > 0$ such that*

$$\max_{i,j\in[N], i\neq j} \big\| q^{(3)}_{ij} \big\|_{L^2(\Omega)} \leq CN^{-3/2}, \quad \max_{i,j\in[N], i\neq j} \big\| q^{(5)}_{ij} \big\|_{L^2(\Omega)} \leq CN^{-3/2}. \tag{3.9}$$

Moreover, for fixed $k, k_1, k_2 \geq 3$, let $\boldsymbol{Q}^{(1)}_{>k_1,k_2}$, $\boldsymbol{Q}^{(1)}_{\leq k_1,k_2}$, $\boldsymbol{Q}^{(2)}_{>k}$, $\boldsymbol{Q}^{(2)}_{\leq k}$, $\boldsymbol{Q}^{(6)}_{>k_1,k_2}$ and $\boldsymbol{Q}^{(2)}_{>k_1,k_2}$ be defined as in (3.7). Then, there exists $C > 0$, independent of N and k, as well as some constant $C'_k > 0$ that depends on k, but that is independent of N so that

$$\begin{aligned} \max_{i,j\in[N]} \big\| q^{(1)}_{>k_1,k_2,ij} \big\|_{L^2(\Omega)} \leq Ce^{-\min(k_1,k_2)\log\beta^{-2}} N^{-1}, \\ \max_{i\in[N]} \big\| q^{(2)}_{>k,ii} \big\|_{L^2(\Omega)} \leq Ce^{-k\log\beta^{-2}} N^{-1/2}, \quad \max_{i\in[N]} \big\| q^{(2)}_{\leq k,ii} \big\|_{L^4(\Omega)} \leq C'_k N^{-1/2}. \end{aligned} \tag{3.10}$$

Moreover, for k_N as defined in (2.10), we have for every $\epsilon > 0$ that

$$\max_{i,j\in[N]:i\neq j} \| q^{(6)}_{\leq k_N^4, k_N^2, ij} \|_{L^2(\Omega)} \leq N^{-3/2+\epsilon}, \quad \lim_{N\to\infty} \mathbb{P}\Big(\max_{i,j\in[N]:i\neq j} | q^{(6)}_{>k_N^4, k_N^2, ij} | > N^{-\log N} \Big) = 0. \tag{3.11}$$

Proof. We prove (3.9), (3.10) and (3.11) for fixed $i, j \in [N]$ with $i \neq j$ and, as becomes clear in the bounds below, all constants turn out to be independent of $i, j \in [N]$.

Let us start with (3.9). Neglecting the prefactor β^2 in $q^{(3)}_{ij}$, we compute

$$\begin{aligned} \mathbb{E}\Big(\sum_{k:k\neq i} g_{kj}^2 \sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}: \\ \{i,j\}\in\gamma,\, k\in\mathcal{V}_\gamma}} \prod_{e\in\gamma:e\neq\{i,j\}} \beta g_e \Big)^2 \leq 2\mathbb{E}\Big(\sum_{k:k\neq i} \beta g_{kj}^3 \sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}: \\ \{i,j\},\{j,k\}\in\gamma}} \prod_{e\in\gamma:e\neq\{i,j\},\{j,k\}} \beta g_e \Big)^2 \\ + 2\mathbb{E}\Big(\sum_{k:k\neq i} g_{kj}^2 \sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:k\in\mathcal{V}_\gamma, \\ \{i,j\}\in\gamma,\{j,k\}\notin\gamma}} \prod_{e\in\gamma:e\neq\{i,j\}} \beta g_e \Big)^2. \end{aligned}$$

Now, using that the odd moments of each weight g_{kj} vanish and that the summation runs over $k \in [N]$ with $k \neq i$, such that a graph that contains $\{i,j\}, \{k,j\} \in \gamma \in \Gamma_{\mathrm{loop}}$

can not contain another edge $\{k',j\}$ for $k' \notin \{i,k\}$, we obtain that

$$\begin{aligned}
\mathbb{E}\Big(\sum_{k:k\neq i} g_{kj}^3 \sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:\\ \{i,j\},\{j,k\}\in\gamma}} \prod_{\substack{e\in\gamma:\\ e\neq\{i,j\},\{j,k\}}} \beta g_e\Big)^2 &= 15N^{-3}\sum_{k:k\neq i}\mathbb{E}\Big(\sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:\\ \{i,j\},\{j,k\}\in\gamma}} \prod_{\substack{e\in\gamma:\\ e\neq\{i,j\},\{j,k\}}} \beta g_e\Big)^2\\
&= 15N^{-3}\sum_{k:k\neq i}\sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:\\ \{i,j\},\{j,k\}\in\gamma}} (\beta^2)^{|\gamma|-2}N^{-|\gamma|+2}\\
&\leq 15N^{-3}\sum_{k:k\neq i}\sum_{l\geq 3}(\beta^2)^{l-2}N^{l-3}N^{-l+2} \leq CN^{-3},
\end{aligned}$$

where we used that there are $(N-3)(N-4)\dots(N-l+1)\leq N^{l-3}$ cycles of length $l\geq 4$ with edges $\{i,j\},\{j,k\}$, for fixed $i,j,k\in[N]$ (if $l=3$, there is a unique such cycle). Similarly, we can use that g_{kj} and the weight of any self-avoiding path from vertex i to vertex j that does not contain the edge $\{k,j\}$ are independent, and that the weights of any two different self-avoiding paths are orthogonal in $L^2(\Omega)$. This yields

$$\begin{aligned}
&\mathbb{E}\Big(\sum_{k:k\neq i,j} g_{kj}^2 \sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:k\in\mathcal{V}_\gamma,\\ \{i,j\}\in\gamma,\{j,k\}\notin\gamma}} \prod_{e\in\gamma:e\neq\{i,j\}} \beta g_e\Big)^2\\
&= \sum_{k:k\neq i,j}\big(\mathbb{E}\,g_{kj}^4\big)\,\mathbb{E}\Big(\sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:k\in\mathcal{V}_\gamma,\\ \{i,j\}\in\gamma,\{j,k\}\notin\gamma}} \prod_{e\in\gamma:e\neq\{i,j\}} \beta g_e\Big)^2\\
&\quad+\mathbb{E}\sum_{\substack{k,k':k\neq i,j;\\ k'\neq i,j;k\neq k'}} g_{kj}^2 g_{k'j}^2 \Big(\sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:k\in\mathcal{V}_\gamma,\\ \{i,j\}\in\gamma,\{j,k\}\notin\gamma}} \prod_{e\in\gamma:e\neq\{i,j\}} \beta g_e\Big)\Big(\sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:k'\in\mathcal{V}_{\gamma'},\\ \{i,j\}\in\gamma',\{j,k'\}\notin\gamma'}} \prod_{e\in\gamma':e\neq\{i,j\}} \beta g_e\Big)\\
&= 3N^{-2}\sum_{k:k\neq i,j}\sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:k\in\mathcal{V}_\gamma,\\ \{i,j\}\in\gamma,\{j,k\}\notin\gamma}} (\beta^2)^{|\gamma|-1}N^{-|\gamma|+1}\\
&\quad+\mathbb{E}\sum_{\substack{k,k':k\neq i,j;\\ k'\neq i,j;k\neq k'}} g_{kj}^2 g_{k'j}^2 \Big(\sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:k\in\mathcal{V}_\gamma,\\ \{i,j\}\in\gamma,\{j,k\}\notin\gamma}} \prod_{e\in\gamma:e\neq\{i,j\}} \beta g_e\Big)\Big(\sum_{\substack{\gamma\in\Gamma_{\mathrm{loop}}:k'\in\mathcal{V}_{\gamma'},\\ \{i,j\}\in\gamma',\{j,k'\}\notin\gamma'}} \prod_{e\in\gamma':e\neq\{i,j\}} \beta g_e\Big).
\end{aligned}$$

Using once again that the odd moments of the edge weights vanish, one can also factorize the expectation in the last term on the r.h.s. of the previous equation to get

$$\begin{aligned}
&\mathbb{E}\Big(\sum_{k:k\neq i} g_{kj}^2 \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:k\in\mathcal{V}_\gamma,\\ \{i,j\}\in\gamma,\{j,k\}\notin\gamma}} \prod_{e\in\gamma:e\neq\{i,j\}} \beta g_e\Big)^2\\
&= 3N^{-2}\sum_{k:k\neq i,j} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:k\in\mathcal{V}_\gamma,\\ \{i,j\}\in\gamma,\{j,k\}\notin\gamma}} (\beta^2)^{|\gamma|-1}N^{-|\gamma|+1} + N^{-2}\sum_{\substack{k,k':k\neq i,j;\\ k'\neq i,j;k\neq k'}} \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:k,k'\in\mathcal{V}_\gamma,\\ \{i,j\}\in\gamma,\{j,k\},\{j,k'\}\notin\gamma}} (\beta^2)^{|\gamma|-1}N^{-|\gamma|+1}\\
&\leq CN^{-2}\sum_{k:k\neq i,j}\sum_{l\geq 3} l(\beta^2)^{l-1}N^{l-3}N^{-l+1} + N^{-2}\sum_{\substack{k,k':k\neq i,j;\\ k'\neq i,j;k\neq k'}} l^2(\beta^2)^{l-1}N^{l-4}N^{-l+1} \leq CN^{-3}.
\end{aligned}$$

Here, we used that there are not more than lN^{l-3} cycles of length l with edge $\{i,j\}$ and some fixed vertex $k\neq i,j$, and not more than l^2N^{l-4} cycles with edge $\{i,j\}$ and two different vertices $k,k'\neq i,j$. In summary, this proves $\|q_{ij}^{(3)}\|_{L^2(\Omega)}\leq CN^{-3/2}$.

Next, using once again the orthogonality of the weights of two different self-avoiding paths in $L^2(\Omega)$, in particular the orthogonality of g_{ij} to every self-avoiding path from vertex i to vertex j of length greater than two, the bound on $q_{ij}^{(5)}$ is obtained as

$$\begin{aligned}
\mathbb{E}\, g_{ij}^4 p_{ij}^2 &= 15\beta^2N^{-3} + 3N^{-2}\sum_{\{i,j\}\in\gamma\in\Gamma_{\text{loop}}} (\beta^2)^{|\gamma|-1}N^{-|\gamma|+1}\\
&\leq 15\beta^2N^{-3} + 3N^{-2}\sum_{l\geq 3}(\beta^2)^{l-1}(N-2)(N-3)\dots(N-l+1)N^{-l+1} \leq CN^{-3}.
\end{aligned}$$

Let us now switch to (3.10), fixing $k,k_1,k_2\geq 3$. The large graph contributions are controlled by using the $L^2(\Omega)$ orthogonality of different cycles. Using the explicit representation

$$\begin{aligned}
q_{>k_1,k_2,ij}^{(1)} = -2 &\sum_{\substack{\gamma_1\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma_1,|\gamma_1|>k_1}} \Big(\prod_{e\in\gamma_1:e\neq\{i,j\}}\beta g_e\Big) \sum_{\substack{\gamma_2\in\Gamma_{\text{loop}}:\\ \mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}=\{j\}}} \prod_{e'\in\gamma_2}\beta g_{e'}\\
-2 &\sum_{\substack{\gamma_1\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma_1,|\gamma_1|\leq k_1}} \Big(\prod_{e\in\gamma_1:e\neq\{i,j\}}\beta g_e\Big) \sum_{\substack{\gamma_2\in\Gamma_{\text{loop}}:|\gamma_2|>k_2,\\ \mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}=\{j\}}} \prod_{e'\in\gamma_2}\beta g_{e'},
\end{aligned}$$

we find that

$$
\begin{aligned}
\big\|q^{(1)}_{>k_1,k_2,ij}\big\|^2_{L^2(\Omega)} \leq C &\sum_{\substack{\gamma_1\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma_1,|\gamma_1|>k_1}} (\beta^2)^{|\gamma_1|-1}N^{-|\gamma_1|+1} \sum_{\substack{\gamma_2\in\Gamma_{\text{loop}}:\\ \mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}=\{j\}}} (\beta^2)^{|\gamma_2|}N^{-|\gamma_2|} \\
&+ C \sum_{\substack{\gamma_1\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma_1,|\gamma_1|\leq k_1}} (\beta^2)^{|\gamma_1|-1}N^{-|\gamma_1|+1} \sum_{\substack{\gamma_2\in\Gamma_{\text{loop}}:|\gamma_2|>k_2\\ \mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}=\{j\}}} (\beta^2)^{|\gamma_2|}N^{-|\gamma_2|} \\
\leq C &\sum_{\substack{\gamma_1\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma_1,|\gamma_1|>k_1}} (\beta^2)^{|\gamma_1|-1}N^{-|\gamma_1|+1} \sum_{\gamma_2\in\Gamma_{\text{loop}}:j\in\mathcal{V}_{\gamma_2}} (\beta^2)^{|\gamma_2|}N^{-|\gamma_2|} \\
&+ C \sum_{\substack{\gamma_1\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma_1}} (\beta^2)^{|\gamma_1|-1}N^{-|\gamma_1|+1} \sum_{\gamma_2\in\Gamma_{\text{loop}}:|\gamma_2|>k_2,j\in\mathcal{V}_{\gamma_2}} (\beta^2)^{|\gamma_2|}N^{-|\gamma_2|} \\
\leq C &\sum_{l>k_1}(\beta^2)^{l-1}N^{l-2}N^{-l+1}\sum_{l\geq 3}(\beta^2)^{l}N^{l-1}N^{-l} \\
&+ C\sum_{l\geq 3}(\beta^2)^{l-1}N^{l-2}N^{-l+1}\sum_{l>k_2}(\beta^2)^{l}N^{l-1}N^{-l} \leq CN^{-2}(\beta^2)^{-\min(k_1,k_2)}.
\end{aligned}
$$

Analogously, we obtain

$$
\big\|q^{(2)}_{>k,ii}\big\|^2_{L^2(\Omega)} \leq C \sum_{\substack{\gamma\in\Gamma_{\text{loop}}:\\ |\gamma|>k,i\in\mathcal{V}_\gamma}} (\beta^2)^{|\gamma_1|}N^{-|\gamma_1|} \leq C\sum_{l>k}(\beta^2)^{l}N^{l-1}N^{-l} \leq CN^{-1}e^{-k\log\beta^{-2}}.
$$

Consider now the $L^4(\Omega)$ estimate on $q^{(2)}_{>k,ii}$. Neglecting constant prefactors, we have that

$$
\mathbb{E}\big(q^{(2)}_{\leq k,ii}\big)^4 \leq \sum_{\substack{\gamma_1,\gamma_2,\gamma_3,\gamma_4\in\Gamma_{\text{loop}}:\\ i\in\mathcal{V}_{\gamma_t},|\gamma_t|\leq k\,\forall\, t}} \mathbb{E}\prod_{e\in\gamma_1,\ldots,\gamma_4}\beta g_e \tag{3.12}
$$

To estimate the right hand side further, we proceed similarly as in [2, Lemma 3.1] and interpret the sum to range over multigraphs $\gamma=\gamma_1\circ\gamma_2\circ\gamma_3\circ\gamma_4$ with $\gamma_t\in\Gamma_{\text{loop}}$, $i\in\mathcal{V}_{\gamma_t}$ and $|\gamma_t|\leq k$, for each $t\in\{1,2,3,4\}$. By Wick's rule, the expectation of the weight

$$
\mathbb{E}\prod_{e\in\gamma=\gamma_1\circ\gamma_2\circ\gamma_3\circ\gamma_4}\beta g_e
$$

does only contribute to the fourth moment if the multiplicity $n_{uv}(\gamma)$ of each edge is even, and since γ consists of four cycles, we have in this case in fact $n_{uv}(\gamma)\in\{0,2,4\}$ for each $u,v\in[N]$ with $u<v$. Moreover, we can bound the contribution of such a γ trivially by

$$
\mathbb{E}\prod_{e\in\gamma}\beta g_e = (\beta^2)^{|\gamma|/2}\prod_{1\leq u<v\leq N}\mathbb{E}\, g_{uv}^{n_{uv}(\gamma)} \leq (3\beta^2)^{2k}N^{-|\gamma|/2}.
$$

Now, notice that the contribution of each multi-graph $\gamma=\gamma_1\circ\gamma_2\circ\gamma_3\circ\gamma_4$ to the sum on the r.h.s. in (3.12) depends only on the occupation numbers of the edges $n_{uv}(\gamma)$. In other words, if there exists a permutation $\pi : [N] \to [N]$ such that $n_{uv}(\gamma) = n_{\pi(u)\pi(v)}(\gamma')$ for all $u < v$ (i.e., γ and γ' are isomorphic multi-graphs), then

$$\mathbb{E} \prod_{e\in\gamma} \beta g_e = \mathbb{E} \prod_{e\in\gamma'} \beta g_e.$$

Defining $\gamma \sim \gamma'$ to be equivalent if and only if there exists a permutation $\pi : [N] \to [N]$ such that $n_{uv}(\gamma) = n_{\pi(u)\pi(v)}(\gamma')$ for all $u < v$, and denoting by $[\gamma]$ the corresponding equivalence classes that partition the set

$$S_4 := \big\{\gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3 \circ \gamma_4 : i \in \mathcal{V}_{\gamma_t}, |\gamma_t| \leq k, \ \forall\, t = 1, 2, 3, 4\big\},$$

into a disjoint union of equivalence classes, we obtain

$$\mathbb{E}\big(q^{(2)}_{\leq k,ii}\big)^4 \leq (3\beta^2)^{2k} \sum_{[\gamma]} |[\gamma]| N^{-|[\gamma]|/2} = (3\beta^2)^{2k} \sum_{[\gamma]} |[\gamma]| e^{-\frac{1}{4}\sum_{i=1}^N n_i([\gamma])}.$$

Now, the number of elements in an equivalence class is bounded by $N^{|\mathcal{V}_{[\gamma]}|-1}$, because all multi-graphs have vertices in $[N]$ and because each graph contains $i \in [N]$, by assumption. This assumption also implies that $n_i([\gamma]) = 8$ for each $[\gamma]$. Together with the fact that the degree $n_i([\gamma]) \geq 4$ for each vertex in $[\gamma]$ (because γ is the product of cycles and every edge must occur at least twice to give a non-zero contribution under $\mathbb{E}$), we find

$$\mathbb{E}\big(q^{(2)}_{\leq k,ii}\big)^4 \leq (3\beta^2)^{2k} N^{-1} \sum_{[\gamma]} N^{|\mathcal{V}_{[\gamma]}| - \frac{1}{4}\sum_{i=1}^N n_i([\gamma])} \leq (3\beta^2)^{2k} N^{-2} \sum_{[\gamma]} \leq (3\beta^2)^{2k} C'_k N^{-2}.$$

Here, we used that the number of multi-graphs with $l \leq 4k$ edges is bounded by some $C'_k > 0$ determined uniquely by k (in particular independent of N). This proves (3.10).

Finally, let us explain the bounds in (3.11). Let us note first the upper bound

$$\mathbb{E}[(q^{(6)}_{\leq k_N^4, k_N^2, ij})^2] \leq 4\mathbb{E}\Bigg(\sum_{\substack{\gamma_1\in\Gamma_{\text{loop}}:\\ \{i,j\}\in\gamma_1, |\gamma_1|\leq k_N^4}} \Big(\prod_{e\in\gamma_1 : e\neq\{i,j\}} \beta g_e \Big) \sum_{\substack{\gamma_2\in\Gamma_{\text{sc}}:|\gamma_2|\leq k_N^4,\\ |\mathcal{V}_{\gamma_1}\cap\mathcal{V}_{\gamma_2}|\geq 2}} \prod_{e'\in\gamma_2} \beta g_{e'}, \Bigg)^2,$$

where we removed the restrictions $j \in \mathcal{V}_{\gamma_2}$, $\gamma_2 \in \Gamma_{\text{loop}}$ and where we relaxed the large graph constraint to $|\gamma_2| \leq k_N^4$, as all cross terms give positive contributions under taking the expectation, by Wick's rule. Now, it is clear that the r.h.s. in the last bound can be treated in the same way as $\mathbb{E}[(R^{(2)}_{ij})^2]$ in Proposition 2.8 (the differences to $R^{(2)}_{ij}$ are only the graph size constraints and the fact that the edge weights $\tanh(\beta g_e)$ are replaced by βg_e; both differences do not affect the arguments in the proof of Proposition 2.8). Thus

$$\mathbb{E}[(q^{(6)}_{\leq k_N^4, k_N^2, ij})^2] \leq CN^{-3+\epsilon}$$

for every $\epsilon > 0$ small enough and N large enough.

Similarly, the contribution $q^{(6)}_{>k_N^4, k_N^2, ij}$ can be controlled like the term $R^{(3)}_{ij}$, defined in (2.8), as in the proof of Lemma 2.5. This yields the probability estimate in (3.11). $\square$

We are now ready to conclude Theorem 1.1.

Proof of Theorem 1.1. By Corollary 2.9, it is enough to show that

$$\|\mathbf{P} - (1+\beta^2-\beta\mathbf{G})^{-1}\|_{\mathrm{op}} \to 0$$

as $N\to\infty$, in the sense of probability. Using $\|\beta\mathbf{G}\|_{\mathrm{op}} < 2\beta + (1-\beta^2)\epsilon/2$, we first bound

$$\|\mathbf{P} - (1+\beta^2-\beta\mathbf{G})^{-1}\|_{\mathrm{op}} \leq \|(1+\beta^2-\beta\mathbf{G})^{-1}\|_{\mathrm{op}}\|\mathbf{Q}\|_{\mathrm{op}} \leq (1-\beta)^{-2}(1+\epsilon)\|\mathbf{Q}\|_{\mathrm{op}}$$

so that, by the decomposition (3.8), we obtain that

$$\begin{aligned}
&\frac{\|\mathbf{P} - (1+\beta^2-\beta\mathbf{G})^{-1}\|_{\mathrm{op}}}{(1-\beta)^{-2}(1+\epsilon)}\\
&\leq \big(\|\mathbf{P} - (1+\beta^2-\beta\mathbf{G})^{-1}\|_{\mathrm{op}} + C\big)\big(\beta^2 N^{-1} + \|\mathbf{Q}^{(2)}\|_{\mathrm{op}} + \|\mathbf{Q}^{(2)}_{>k_2}\|_{\mathrm{op}} + \|\mathbf{Q}^{(4)}\|_{\mathrm{op}}\big)\\
&\quad + \|\mathbf{Q}^{(3)}\|_{\mathrm{op}} + \|\mathbf{Q}^{(5)}\|_{\mathrm{op}} + \|\mathbf{Q}^{(1)}_{>k_1,k_2}\|_{\mathrm{op}} + \|\mathbf{Q}^{(2)}_{>k_2}\|_{\mathrm{op}} + \|\mathbf{Q}^{(6)}_{>k_1,k_2}\|_{\mathrm{op}} + \|\mathbf{Q}^{(6)}_{\leq k_1,k_2}\|_{\mathrm{op}}
\end{aligned} \tag{3.13}$$

for some $C = C_\beta > 0$ and for fixed $k_1, k_2 \geq 3$. Choosing $k_1 = k_N^4, k_2 = k_N^2$, defined in (2.10), s.t. $k_N \to \infty$ as $N\to\infty$, we make sure that the k_1- and k_2-dependent matrices on the r.h.s. converge to zero, in the sense of probability. Indeed, by (3.11), we get

$$\mathbb{P}\Big(\|\mathbf{Q}^{(6)}_{>k_N^4,k_N^2}\|_{\mathrm{op}} > \delta\Big) \leq \mathbb{P}\Big(\|\mathbf{Q}^{(6)}_{>k_N^4,k_N^2}\|_{\mathrm{F}} > \delta\Big) \leq \mathbb{P}\Big(\max_{i,j\in[N]:i\neq j} |q^{(6)}_{>k_N^4,k_N^2,ij}| > \delta N^{-1}\Big) \to 0$$

as $N\to\infty$, for every $\delta>0$. Similarly, we obtain from (3.10) and (3.11) that

$$\begin{aligned}
&\mathbb{P}\Big(\|\mathbf{Q}^{(1)}_{>k_N^4,k_N^2}\|_{\mathrm{op}} > \delta\Big) + \mathbb{P}\Big(\|\mathbf{Q}^{(2)}_{>k_N^2}\|_{\mathrm{op}} > \delta\Big) + \mathbb{P}\Big(\|\mathbf{Q}^{(6)}_{\leq k_N^4,k_N^2}\|_{\mathrm{op}} > \delta\Big)\\
&\leq \mathbb{P}\Big(\|\mathbf{Q}^{(1)}_{>k_N^4,k_N^2}\|_{\mathrm{F}} > \delta\Big) + \mathbb{P}\Big(\|\mathbf{Q}^{(2)}_{>k_N^2}\|_{\mathrm{F}} > \delta\Big) + \mathbb{P}\Big(\|\mathbf{Q}^{(6)}_{\leq k_N^4,k_N^2}\|_{\mathrm{F}} > \delta\Big)\\
&\leq \delta^{-2}N^2 \max_{i,j\in[N]:i\neq j}\Big(\|q^{(1)}_{>k_N^4,k_N^2,ij}\|^2_{L^2(\Omega)} + \|q^{(2)}_{>k_N^2,ij}\|^2_{L^2(\Omega)} + \|q^{(6)}_{\leq k_N^4,k_N^2,ij}\|^2_{L^2(\Omega)}\Big) \leq \delta^{-2}N^{-1+\epsilon}
\end{aligned}$$

for fixed $\epsilon>0$ small enough, so that

$$\lim_{N\to\infty}\|\mathbf{Q}^{(1)}_{>k_N^4,k_N^2}\|_{\mathrm{op}} = \lim_{N\to\infty}\|\mathbf{Q}^{(2)}_{>k_N^2}\|_{\mathrm{op}} = \lim_{N\to\infty}\|\mathbf{Q}^{(6)}_{\leq k_N^4,k_N^2}\|_{\mathrm{op}} = \lim_{N\to\infty}\|\mathbf{Q}^{(6)}_{>k_N^4,k_N^2}\|_{\mathrm{op}} = 0.$$

Inserting this into (3.13), we conclude that

$$\begin{aligned}
&\frac{\|\mathbf{P} - (1+\beta^2-\beta\mathbf{G})^{-1}\|_{\mathrm{op}}}{(1-\beta)^{-2}(1+\epsilon)}\\
&\leq \big(\|\mathbf{P} - (1+\beta^2-\beta\mathbf{G})^{-1}\|_{\mathrm{op}} + C\big)\big(\|\mathbf{Q}^{(2)}\|_{\mathrm{op}} + \|\mathbf{Q}^{(4)}\|_{\mathrm{op}} + o_1(1)\big)\\
&\quad + \|\mathbf{Q}^{(3)}\|_{\mathrm{op}} + \|\mathbf{Q}^{(5)}\|_{\mathrm{op}} + o_2(1)
\end{aligned} \tag{3.14}$$

for two errors $o_1(1)$ and $o_2(1)$ that converge $\lim_{N\to\infty} o_1(1) = \lim_{N\to\infty} o_2(1) = 0$ in probability. Now, consider the remaining matrices $\mathbf{Q}^{(2)}$, $\mathbf{Q}^{(3)}$, $\mathbf{Q}^{(4)}$ and $\mathbf{Q}^{(5)}$. Using that $\|\mathbf{Q}^{(3)}\|_{\mathrm{op}} \le \|\mathbf{Q}^{(3)}\|_{\mathrm{F}}$, we obtain with the bounds (3.9) for every $\delta > 0$ that

$$\mathbb{P}\big(\|\mathbf{Q}^{(3)}\|_{\mathrm{op}} > \delta\big) \le \mathbb{P}\big(\|\mathbf{Q}^{(3)}\|_{\mathrm{F}} > \delta\big) \le \delta^{-2} N^2 \max_{i,j\in[N]:i\neq j} \|q^{(3)}_{ij}\|^2_{L^2(\Omega)} \le C\delta^{-2}N^{-1}$$

so that $\lim_{N\to\infty} \|\mathbf{Q}^{(3)}\|_{\mathrm{op}} = 0$. The same argument implies $\lim_{N\to\infty} \|\mathbf{Q}^{(5)}\|_{\mathrm{op}} = 0$ and for $\mathbf{Q}^{(4)}$, we notice that, by a standard concentration and union bound, we have

$$\mathbb{P}\Big(\max_{i\in[N]} \Big|\sum_u \big(g_{iu}^2 - \mathbb{E}\, g_{iu}^2\big)\Big| > \delta\Big) \le Ce^{-cN\delta^2}$$

for suitable $C, c > 0$ that are independent of $N \in \mathbb{N}$ and $\delta > 0$. Recalling the definition of $q^{(4)}_{ii}$ in (3.6), this trivially implies that in the sense of probability we have

$$\lim_{N\to\infty} \|\mathbf{Q}^{(4)}\|_{\mathrm{op}} = \beta^2 \lim_{N\to\infty} \max_{i\in[N]} \Big|\sum_u \big(g_{iu}^2 - \mathbb{E}\, g_{iu}^2\big)\Big| = 0.$$

Finally, consider $\mathbf{Q}^{(2)}$. For $\delta > 0$ and fixed $k \ge 3$, we use (3.10) to estimate

$$\begin{aligned}
\mathbb{P}\big(\|\mathbf{Q}^{(2)}\|_{\mathrm{op}} > \delta\big) &\le \mathbb{P}\big(\|\mathbf{Q}^{(2)}_{>k}\|_{\mathrm{op}} > \delta/2\big) + \mathbb{P}\big(\|\mathbf{Q}^{(2)}_{\le k}\|_{\mathrm{op}} > \delta/2\big)\\
&\le \delta^{-2} N \max_{i\in[N]} \|q^{(2)}_{>k,ii}\|^2_{L^2(\Omega)} + \delta^{-4} N \max_{i\in[N]} \|q^{(2)}_{>k,ii}\|^4_{L^4(\Omega)}\\
&\le C\delta^{-2} e^{-k\log\beta^{-2}} + C_k'\delta^{-4}N^{-1}.
\end{aligned}$$

Sending first $N\to\infty$ and then $k\to\infty$, we conclude that $\lim_{N\to\infty} \|\mathbf{Q}^{(2)}\|_{\mathrm{op}} = 0$.

Collecting the bounds from above and inserting them into (3.14), we conclude that

$$\|\mathbf{P} - (1+\beta^2 - \beta\mathbf{G})^{-1}\|_{\mathrm{op}} \le \frac{o_1'(1)}{(1-\beta)^2(1+\epsilon)^{-1} - o_2'(1)}$$

for two errors $o_1'(1)$ and $o_2'(1)$ that satisfy $\lim_{N\to\infty} o_1'(1) = \lim_{N\to\infty} o_2'(1) = 0$. In particular, we conclude that $\lim_{N\to\infty} \|\mathbf{P} - (1+\beta^2-\beta\mathbf{G})^{-1}\|_{\mathrm{op}} = 0$ in probability. □

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