# ON THE CONCAVITY OF THE TAP FREE ENERGY IN THE SK MODEL

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Abstract. We analyse the Hessian of the Thouless-Anderson-Palmer (TAP) free energy for the Sherrington-Kirkpatrick model, below the de Almeida-Thouless line, evaluated in Bolthausen's approximate solutions of the TAP equations. We show that the empirical spectral distribution weakly converges to a measure with negative support below the AT line, and that the support includes zero on the AT line. In this "macroscopic" sense, we show that TAP free energy is concave in the order parameter of the theory, i.e. the random spin-magnetisations. This proves a spectral interpretation of the AT line. We also find different magnetizations than Bolthausen's approximate solutions at which the Hessian of the TAP free energy has positive outlier eigenvalues. In particular, when the magnetizations are assumed to be independent of the disorder, we prove that Plefka's second condition is equivalent to all eigenvalues being negative. On this occasion, we extend the convergence result of Capitaine et al. (Electron. J. Probab. **16**, no. 64, 2011) for the largest eigenvalue of perturbed complex Wigner matrices to the GOE.

Contents



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# 1. Introduction

We consider the standard Sherrington-Kirkpatrick (SK for short) model with an external field. In its random Hamiltonian

$$H_{\beta,h}(\sigma) := \frac{\beta}{\sqrt{2N}} \sum_{i,j=1}^{N} g_{ij}\sigma_i\sigma_j + h\sum_{i=1}^{N}\sigma_i \tag{1.1}$$

for $N \in \mathbb{N}$ spins $\sigma = (\sigma_i) \in \Sigma_N := \{-1,1\}^N$, the disorder is modeled by i.i.d. centered Gaussians $g_{ij}$ with variance 1 on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The parameters $\beta > 0$ and $h \in \mathbb{R}$ are called inverse temperature and external field. The partition function is given by

$$Z_N(\beta,h) := 2^{-N}\sum_{\sigma} \exp H_{\beta,h}(\sigma), \tag{1.2}$$

and the free energy by

$$f_N(\beta,h) := \frac{1}{N}\log Z_N(\beta,h). \tag{1.3}$$

A well-known consequence of Gaussian concentration of measure is that the free energy is self-averaging in the sense that

$$f(\beta,h) := \lim_{N\to\infty}\frac{1}{N}\log Z_N(\beta,h) = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_N(\beta,h) \text{ almost surely.} \tag{1.4}$$

The existence of the limit on the right-hand side was established in a celebrated paper by Guerra and Toninelli [20]. The limit is given by the Parisi variational formula (see [32, 25]). In high temperature ($\beta$ small), $f(\beta,h)$ is also given by the replica-symmetric formula, originally proposed by Sherrington and Kirkpatrick [29]:

**Theorem 1.1** ([29, 8, 10])**.** *There exists $\beta_0 > 0$ such that for all $h$, $\beta$ with $0 < \beta \le \beta_0$,*

$$f(\beta,h) = RS(\beta,h) := \inf_{q\ge 0}\left\{\mathsf{E}\log\cosh(h+\beta\sqrt{q}Z) + \frac{\beta^2(1-q)^2}{4}\right\}, \tag{1.5}$$

*where $Z$ is a standard Gaussian.*

Guerra [19] (see also Talagrand [31, Proposition 1.3.8] where an independent proof of Latala is also mentioned) proved that for $h \neq 0$, the infimum is uniquely attained at $q = q(\beta,h)$ which satisfies

$$q = \mathsf{E}\tanh^2(h+\beta\sqrt{q}Z). \tag{1.6}$$

Here and in the following, $Z$ (under a probability $\mathsf{P}$ with associated expectation $\mathsf{E}$) always denotes a standard Gaussian. For $h \neq 0$, the fixed point equation (1.6) has a unique solution which we denote in the sequel by $q$. A proof of Theorem 1.1 based on an approach of Thouless-Anderson-Palmer (TAP for short) [35] can be found in [8]. The critical temperature $\beta_0$ in Theorem 1.1 has then been improved in [10] using the same approach. Actually, $f(\beta,h) = RS(\beta,h)$ is believed to hold under the de Almeida-Thouless condition (AT for short), i.e. for $(\beta,h)$ with

$$\beta^2\mathsf{E}\frac{1}{\cosh^4(h+\beta\sqrt{q}Z)} \le 1, \tag{1.7}$$

but this problem is still open (however, Toninelli [36] proved that when (1.7) is not satisfied, then the assertion of Theorem 1.1 does not hold anymore). De Almeida and Thouless found the condition (1.7) in 1978 in the context of an instability in the replica procedure [2] which is hard to make rigorous. We also mention that Chen [12] recently established the de Almeida-Thouless line as the transition curve between the replica symmetric and the replica symmetry breaking phases in a SK model with centered Gaussian external field.

To state our results, we first introduce the TAP free energy of the SK model. Analysis of the SK model in terms of the TAP equations was first given by [35]: shortening $\bar{g}_{ij} = \frac{1}{\sqrt{2}}(g_{ij} + g_{ji})$, and for $\boldsymbol{m} = (m_i) \in [-1,1]^N$, this is given by

$$\mathrm{TAP}_N(\boldsymbol{m}) = \frac{\beta}{\sqrt{N}} \sum_{\substack{i,j=1 \\ i<j}}^{N} \bar{g}_{ij} m_i m_j + h \sum_{i=1}^{N} m_i + \frac{\beta^2}{4} N \left(1 - \frac{1}{N} \sum_{i=1}^{N} m_i^2\right)^2 - \sum_{i=1}^{N} I(m_i), \quad (1.8)$$

where for $x \in [-1,1]$,

$$I(x) = \frac{1+x}{2} \log(1+x) + \frac{1-x}{2} \log(1-x) = x \tanh^{-1}(x) - \log\cosh\tanh^{-1}(x)\,. \quad (1.9)$$

The TAP free energy can be related to the free energy by a variational principle: Chen and Panchenko [13, Theorem 1] show that

$$f(\beta, h) = \lim_{\epsilon\downarrow 0} \lim_{N\to\infty} \mathbb{E} \max N^{-1} \mathrm{TAP}_N(\boldsymbol{m}), \quad (1.10)$$

where the maximum is over all $\boldsymbol{m} \in [-1,1]^N$ with $N^{-1}\sum_{i=1}^N m_i^2 \in [q_P - \epsilon, q_P + \epsilon]$, $q_P$ denoting the right edge of the support of the Parisi measure. We also mention that an upper bound of the free energy in terms of the TAP free energy has recently been given by Belius [4]. For the SK model with spherical spins, a variational principle for the TAP free energy has been proved in [5].

The TAP free energy can also be interpreted in terms of a power expansion up to second order of the Gibbs potential of the SK model [27] (see Appendix A and also [21] for further discussion). A necessary condition of Plefka [27] for the convergence of the power expansion is that the magnetizations are in

$$P_N^1 := \{\boldsymbol{m} \in [-1,1]^N, \frac{\beta^2}{N} \sum_{i=1}^{N} \left(1 - m_i^2\right)^2 < 1\}. \quad (1.11)$$

$P_N^1$ is the set of magnetizations satisfying the so-called first Plefka condition. Before Plefka, this condition was also noted by Bray and Moore [9] who investigated the Hessian matrix of the TAP free energy. For the stability of a diagrammatic expansion of the free energy, Sommers [30] also obtained condition (1.11). There is no rigorous justification whether the first Plefka condition suffices for neglecting the higher-order terms (cf. also the discussion in [24]).

From Plefka's expansion, it is reasonable to expect concavity of $\mathrm{TAP}_N$ in $\boldsymbol{m}$ if these higher-order terms and further correction terms can be neglected. Indeed, as we discuss in Appendix A, the free energy can be represented as

$$F_N(\beta, h) = \sup_{\boldsymbol{m}\in[-1,1]^N} \left\{\mathrm{TAP}_N(\boldsymbol{m}) + R(\boldsymbol{m})\right\}, \quad (1.12)$$

where the expression over which the supremum is taken turns out to be a Legendre transform and thus must be concave. If the Hessian of all correction terms which are subsumed in $R(\boldsymbol{m})$ vanishes, then also $\mathrm{TAP}_N$ has to be concave. Let us remark that the TAP functional is not necessarily concave: we consider the Hessian

$$\boldsymbol{H}(\boldsymbol{m}) := \frac{\partial^2}{\partial m_i \, \partial m_j} \mathrm{TAP}_N(\boldsymbol{m}) \tag{1.13}$$

at arbitrary magnetizations $\boldsymbol{m} \in [-1,1]^N$. Denoting by $\lambda_1(\boldsymbol{M})$ the largest eigenvalue of a real and symmetric matrix $\boldsymbol{M}$, we then have:

**Theorem 1.2.** *There exists $\beta_0 \in (0,1)$ such that for all $\beta > \beta_0$, $h \neq 0$, there exist $\epsilon > 0$ and random $\boldsymbol{m}_N \in P_N^1$ such that*

$$\lim_{N\to\infty} \mathbb{P}\left(\lambda_1\left(\boldsymbol{H}(\boldsymbol{m}_N)\right) > \epsilon\right) = 1. \tag{1.14}$$

This observation is proved in Section 8. Now the question arises whether concavity is also lost in the vicinity of the maximizer of $\mathrm{TAP}_N$, as the magnetization $\boldsymbol{m}_N$ for which Theorem 1.2 can be proved is somewhat arbitrary (see (8.1)) and might not be in the domain over which the maximum is taken in the variational principle (1.10).

The fixed points of the TAP equations [35]

$$m_i = \tanh\left(h + \frac{\beta}{\sqrt{N}} \sum_{\substack{j=1\\ j\neq i}}^N \bar{g}_{ij} m_j - \beta^2\left(1 - \frac{1}{N}\sum_{j=1}^N m_j^2\right) m_i\right), \quad i = 1,\ldots,N \tag{1.15}$$

are the critical points of the TAP free energy $\mathrm{TAP}_N$. As we are not able to control these fixed points, we base our analysis on Bolthausen's algorithm [7, 8] which yields a sequence $\boldsymbol{m}^{(k)} \in [-1,1]^N$ of magnetizations that are considered as an approximation of the solutions of (1.15). In [7], the magnetizations $\boldsymbol{m}^{(k)}$ are constructed by a two-step Banach algorithm: $m_i^{(0)} := 0$, $m_i^{(1)} := \sqrt{q}$, and then iteratively

$$m_i^{(k+1)} = \tanh\left(h + \frac{\beta}{\sqrt{N}} \sum_{\substack{j=1\\ j\neq i}}^N \bar{g}_{ij} m_j^{(k)} - \beta^2(1-q) m_i^{(k-1)}\right), \tag{1.16}$$

for $k \geq 1$. In the present paper, we use the similar algorithm from [8] whose precise definition we recall in Section 2. Bolthausen [7, 8] proves that such sequence of magnetisations converges, in the sense of (1.17) below, up to the AT-line. Precisely, by means of a sophisticated conditioning procedure which will be recalled in Section 2, Bolthausen shows that the iterates satisfy

$$\lim_{k,l\to\infty} \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \mathbb{E}\left[\left(m_i^{(k)} - m_i^{(l)}\right)^2\right] = 0, \tag{1.17}$$

*provided* $(\beta, h)$ satisfy the AT-condition. Under a high-temperature condition, Chen and Tang [14, Theorem 3] obtain that

$$\lim_{k\to\infty} \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left[\left(\langle\sigma_i\rangle - m_i^{(k)}\right)^2\right] = 0, \tag{1.18}$$

where $\langle\sigma_i\rangle$ denotes the Gibbs average of $\sigma_i$ under the Hamiltonian (1.1). It is crucial to emphasize that due to the factor $1/N$ in the distance (1.17) and the limit $N \uparrow +\infty$ first, and only in a second step $k, l \uparrow +\infty$, it is not clear whether the convergence of Bolthausen's approximate solutions is sufficiently strong for the interpretation of our results. Notwithstanding, the following suggests that Bolthausen's magnetizations are *good enough* when it comes to computing the limiting free energy within the TAP approximation:

**Theorem 1.3.** *For all $\beta > 0$, $h \neq 0$ satisfying* (1.7)*, the TAP free energy evaluated at the Bolthausen approximate fixed points converges to the replica symmetric functional,*

$$\lim_{k\to\infty} \lim_{N\to\infty} N^{-1}\mathrm{TAP}_N(\boldsymbol{m}^{(k)}) = RS(\beta, h) \text{ in } L_1(\mathbb{P}). \tag{1.19}$$

By the above, we shall henceforth refer to Bolthausen's magnetisations as *approximate solutions* of the TAP-equations. We have not found the proof of Theorem 1.3 in the literature and the proof will be given in Section 3. Véronique Gayrard [17] is going to publish the almost sure convergence in Theorem 1.3. Under the AT condition (1.7), Bolthausen's magnetizations $\boldsymbol{m}^{(k)}$ actually satisfy Plefka's first condition (1.11) with high probability as $N \to \infty$: indeed, it follows from Lemma 2.1 below that

$$\lim_{N\to\infty} \frac{\beta^2}{N}\sum_{i=1}^{N}\left(1 - m_i^{(k)2}\right)^2 = \beta^2 \mathsf{E}\frac{1}{\cosh^4(h+\beta\sqrt{q}Z)} \quad \text{in } L_1(\mathbb{P})\,. \tag{1.20}$$

As a consequence, if the AT condition (1.7) holds with strict inequality, then with probability tending to 1 as $N \to \infty$, Bolthausen's approximate solution satisfies the first Plefka condition, $\boldsymbol{m}^{(k)} \in P_N^1$. That the AT condition and the first Plefka condition are related for suitable magnetizations was clear to Plefka [27].

We now investigate the concavity of the $\mathrm{TAP}_N(\boldsymbol{m})$ functional in the Bolthausen magnetizations, that is, we study the Hessian $\boldsymbol{H}^{(k)}$ of the TAP free energy evaluated in $\boldsymbol{m}^{(k)}$,

$$\boldsymbol{H}^{(k)} := \frac{\partial^2}{\partial m_i\,\partial m_j}\mathrm{TAP}_N(\boldsymbol{m})\Big|_{\boldsymbol{m}=\boldsymbol{m}^{(k)}}. \tag{1.21}$$

We consider the weak limit of the empirical distribution of the eigenvalues $\lambda_i(\boldsymbol{H}^{(k)})$ ($i = 1, \dots, N$) of $\boldsymbol{H}^{(k)}$,

$$\mu_{\boldsymbol{H}^{(k)}} = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i(\boldsymbol{H}^{(k)})}, \tag{1.22}$$

which we show to be concentrated strictly below 0 if the AT condition (1.7) holds with strict inequality, and to touch zero if (1.7) holds with equality. This gives a spectral interpretation of the AT line. Similar observations have been made non-rigorously in [1] (in the $N \to \infty$ limit), and are contained implicitly in [27] (through relation (1.20)).

**Theorem 1.4.** *For all $\beta > 0$, $h \neq 0$ satisfying condition* (1.7)*, the empirical spectral distribution $\mu_{\boldsymbol{H}^{(k)}}$ converges weakly in distribution as $N \to \infty$ followed by $k \to \infty$ to a deterministic limiting measure $\mu$. If* (1.7) *holds with strict inequality, then $\mu(t, \infty) = 0$ for some $t < 0$. If* (1.7) *holds with equality, then* $\sup\{t \in \mathbb{R} : \mu(t, \infty) > 0\} = 0$.

For the proof of Theorem 1.4, we use in Section 5 the explicit control of the weak independence between the disorder $(\bar{g}_{ij})$ and the approximate magnetizations $(m_i^{(k)})$, which is given by Bolthausen's algorithm, and we conclude in Section 6 using results from free probability which we recall in Section 4. We remark, however, that Theorem 1.4 and its proof also pass through for magnetizations $(m_i)$ that are assumed to be independent (or sufficiently weakly dependent) of $(\bar{g}_{ij})$. Theorem 1.4 ensures that under the AT condition, no positive proportion of the eigenvalues of $\boldsymbol{H}^{(k)}$ becomes positive, in this sense, $\boldsymbol{H}^{(k)}$ does not lose concavity "on a macroscopic scale". If and only if the AT condition holds with *strict* inequality, the right edge of the support of the weak limit of the spectrum is strictly smaller than zero, ensuring that *strict* concavity is not lost "on a macroscopic scale". However, we are unable to show that outlier eigenvalues, which are too few to have positive mass and thus are not visible in the weak limit, do not lead to a loss of concavity "on a microscopic scale" for large $N$, $k$. Instead, we prove a rigorous interpretation of Plefka's second condition:

Besides condition $P_N^1$, which is related to the weak limit of the spectrum, Plefka [27] states a second condition

$$P_N^2 := \{\boldsymbol{m} \in [-1,1]^N, \frac{2\beta^2}{N}\sum_{i=1}^N (m_i^2 - m_i^4) < 1\} \tag{1.23}$$

which he relates to the loss of concavity on a "microscopic" scale. However, Owen [24] argues that Plefka's second condition is not necessary and comes from an incorrect assumption, namely that the disorder $(\bar{g}_{ij})$ and the magnetizations $(m_i)$ were independent. Other than for the weak limit of the spectrum in Theorem 1.4, we expect that weak dependence (as present in Bolthausen's "approximate solutions") between $(\bar{g}_{ij})$ and $(m_i)$ does change the condition for the limiting largest eigenvalue to be positive. To illustrate the role of Plefka's second condition, we consider in the following theorem magnetizations $\boldsymbol{m}_N$ that differ from Bolthausen's approximate solutions as they are assumed to be independent of the disorder $\bar{\boldsymbol{g}}$. In this setting, we show rigorously that Plefka's second condition is equivalent to all outlier eigenvalues of the Hessian to be negative. For $\epsilon > 0$, we denote by

$$\bar{P}_N^{2,\epsilon} := \{\boldsymbol{m} \in [-1,1]^N, \frac{2\beta^2}{N}\sum_{i=1}^N (m_i^2 - m_i^4) > 1 + \epsilon\} \tag{1.24}$$

sets of magnetizations that do not satisfy Plefka's second condition.

**Theorem 1.5.** *Let $\beta > 0$, $h \neq 0$ such that the AT condition* (1.7) *is satisfied with strict inequality. Let $\boldsymbol{m}_N$ be $[0,1]^N$-valued random vectors that are independent of $\bar{\boldsymbol{g}}$ and satisfy $\frac{1}{N}\sum_{i=1}^N \delta_{m_{N,i}} \xrightarrow{w} \mathcal{L}(\tanh(h + \beta\sqrt{q}Z))$ as $N \to \infty$, where $Z$ is a standard Gaussian random variable.*

*(i) If $\boldsymbol{m}_N$ takes values only in $P_N^1 \cap P_N^2$, then*

$$\lim_{N\to\infty} \mathbb{P}\left(\lambda_1\left(\boldsymbol{H}\left(\boldsymbol{m}_N\right)\right) \leq 0\right) = 1. \tag{1.25}$$

*(ii) Conversely, if $\boldsymbol{m}_N$ takes values only in $P_N^1 \cap \bar{P}_N^{2;\epsilon}$ for some $\epsilon > 0$, then there exists $\epsilon' > 0$ such that*

$$\lim_{N\to\infty} \mathbb{P}\left(\lambda_1\left(\boldsymbol{H}\left(\boldsymbol{m}_N\right)\right) \geq \epsilon'\right) = 1. \tag{1.26}$$

The proof of Theorem 1.5 is given in Section 7 and relies on a generalization of the convergence result of Capitaine et al. [11] for the largest eigenvalue of perturbed complex Wigner matrices which we state in Lemma 4.4 below.

**Remark 1.6.** *As we may use that $\frac{1}{N}\sum_{i=1}^N \delta_{m_{N,i}} \xrightarrow{w} \mathcal{L}(\tanh(h+\beta\sqrt{q}Z))$ in Theorem 1.5, it follows that $\boldsymbol{m}_N \in P_N^1$ for all sufficiently large $N$ if $\beta > 0$, $h \neq 0$ satisfy the AT condition* (1.7) *with strict inequality. Analogously, it follows that $\boldsymbol{m}_N \in \bar{P}_N^{2;\epsilon}$ for all sufficiently large $N$ if $\beta > 0$, $h \neq 0$ satisfy*

$$2\beta^2 \mathsf{E}\left(\tanh^2(h+\beta\sqrt{q}Z) - \tanh^4(h+\beta\sqrt{q}Z)\right) > 1+\epsilon. \tag{1.27}$$

*Similarly, we have $\boldsymbol{m}_N \in P_N^2$ for all sufficiently large $N$ if $\beta > 0$, $h \neq 0$ are such that*

$$2\beta^2 \mathsf{E}\left(\tanh^2(h+\beta\sqrt{q}Z) - \tanh^4(h+\beta\sqrt{q}Z)\right) < 1. \tag{1.28}$$

*It can be seen numerically that the set of $(\beta, h)$ which satisfy both* (1.7) *and* (1.27) *for some $\epsilon > 0$ is non-empty.*

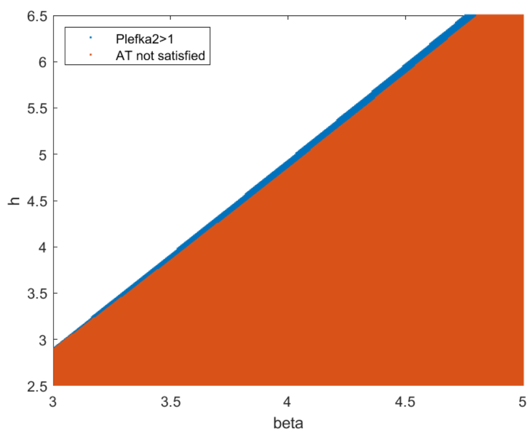


FIGURE 1. Phase diagram of inverse temperature $\beta > 0$ and external field $h > 0$. The region in which the AT condition (1.7) is not satisfied is depicted in red. The region in which (1.7) holds but (1.28) does not hold is depicted in blue.

We remark that Gayrard (personal communication) will actually prove a result complementary to Theorem 1.5, namely the strict concavity of the Hessian (i.e., the maximum eigenvalue is negative) for a specific region of the $\boldsymbol{m}$'s which comprises the Bolthausen approximations, and for $(\beta, h)$ in a region that does not correspond to the AT condition.

## 2. Bolthausen's iterative procedure

We now recall the algorithm from [8] which we will use throughout the paper. Also throughout the paper, we will assume that $\beta > 0$ and $h \neq 0$. A scalar product on $\mathbb{R}^N$ is given by $\langle \boldsymbol{x}, \boldsymbol{y}\rangle := N^{-1}\sum_{i=1}^N x_i y_i$ with associated norm $\|\boldsymbol{x}\| := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}$. Furthermore,

$\boldsymbol{x} \otimes \boldsymbol{y} := N^{-1}(x_i y_j)_{ij}$, and for a matrix $\boldsymbol{A} \in \mathbb{R}^{N\times N}$, we denote its symmetrization by $\bar{\boldsymbol{A}} := \frac{1}{\sqrt{2}}(\boldsymbol{A} + \boldsymbol{A}^T)$.

Let $\boldsymbol{g} = (g_{ij})_{i,j=1,\ldots,N}$ be an array of independent centered Gaussians with variance 1. The interaction matrix will be its symmetrization $\bar{\boldsymbol{g}}$, normalized by $N^{-1/2}$. Let $\psi : [0,q] \to [0,q]$ be defined by

$$\psi(t) = \mathsf{E} \tanh\left(h + \beta\sqrt{t}Z + \beta\sqrt{q-t}Z'\right) \tanh\left(h + \beta\sqrt{t}Z + \beta\sqrt{q-t}Z''\right), \tag{2.1}$$

where $Z, Z', Z''$ are independent standard Gaussians. Then set

$$\gamma_1 := \mathsf{E} \tanh\left(h + \beta\sqrt{q}Z\right), \quad \rho_1 := \sqrt{q}\gamma_1 \tag{2.2}$$

and

$$\rho_k := \psi(\rho_{k-1}), \quad \gamma_k := \frac{\rho_k - \sum_{j=1}^{k-1}\gamma_j^2}{\sqrt{q - \sum_{j=1}^{k-1}\gamma_j^2}}. \tag{2.3}$$

Let $\boldsymbol{g}^{(1)} := \boldsymbol{g}$, $\phi^{(1)} = \mathbf{1}$, $\boldsymbol{m}^{(1)} = \sqrt{q}\mathbf{1}$. With the shorthand $\Gamma_k^2 := \sum_{j=1}^k \gamma_j^2$, we set recursively for $k \in \mathbb{N}$

$$\xi^{(k)} = \tfrac{1}{\sqrt{N}}\boldsymbol{g}^{(k)}\phi^{(k)}, \quad \eta^{(k)} = \tfrac{1}{\sqrt{N}}\boldsymbol{g}^{(k)^T}\phi^{(k)}, \quad \zeta^{(k)} = \frac{1}{\sqrt{2}}\left(\xi^{(k)} + \eta^{(k)}\right), \tag{2.4}$$

$$\boldsymbol{h}^{(k+1)} = h\mathbf{1} + \beta\sum_{s=1}^{k-1}\gamma_s\zeta^{(s)} + \beta\sqrt{q - \Gamma_{k-1}^2}\zeta^{(k)}, \tag{2.5}$$

$$\boldsymbol{m}^{(k+1)} = \tanh(\boldsymbol{h}^{(k+1)}), \tag{2.6}$$

moreover $\{\phi^{(1)}, \ldots, \phi^{(k+1)}\}$ as the Gram-Schmidt orthonormalization of $\{\boldsymbol{m}^{(1)}, \ldots, \boldsymbol{m}^{(k+1)}\}$,

$$\phi^{(k+1)} = \frac{\boldsymbol{m}^{(k+1)} - \sum_{s=1}^k\langle\phi^{(s)}, \boldsymbol{m}^{(k+1)}\rangle\phi^{(s)}}{\left\|\boldsymbol{m}^{(k+1)} - \sum_{s=1}^k\langle\phi^{(s)}, \boldsymbol{m}^{(k+1)}\rangle\phi^{(s)}\right\|}, \tag{2.7}$$

and the modifications of the interaction matrices

$$\boldsymbol{g}^{(k+1)} = \boldsymbol{g}^{(k)} - \sqrt{N}\rho^{(k)}, \tag{2.8}$$

where

$$\rho^{(k)} = \xi^{(k)} \otimes \phi^{(k)} + \phi^{(k)} \otimes \eta^{(k)} - \left\langle\phi^{(k)}, \xi^{(k)}\right\rangle\left(\phi^{(k)} \otimes \phi^{(k)}\right). \tag{2.9}$$

By Lemma 2b of [8], we have

$$\sum_{s=1}^{\infty}\gamma_s^2 = q. \tag{2.10}$$

Noting that $\{\phi^{(s)}\}_{s\le k}$ are orthonormal with respect to $\langle\cdot,\cdot\rangle$, we define

$$P_{ij}^{(k)} = \frac{1}{N}\sum_{s=1}^k \phi_i^{(s)}\phi_j^{(s)}, \tag{2.11}$$

and one readily checks that $\boldsymbol{P}^{(k)}$ is an orthogonal projection. Furthermore, let

$$\mathcal{G}_k = \sigma\left(\xi^{(m)}, \eta^{(m)} : m \le k\right). \tag{2.12}$$

Then $\zeta^{(k)}$ is $\mathcal{G}_k$-measurable and $\boldsymbol{m}^{(k)}$ is $\mathcal{G}_{k-1}$-measurable. Moreover, by Proposition 4 of [8], $\boldsymbol{g}^{(k)}$ is centered Gaussian under $\mathbb{P}(\cdot \mid \mathcal{G}_{k-1})$ with covariances given by

$$V^{(k)}_{ij,st} := \mathbb{E}\Big(g^{(k)}_{ij} g^{(k)}_{st} \,\Big|\, \mathcal{G}_{k-1}\Big) = Q^{(k-1)}_{is} Q^{(k-1)}_{jt}, \tag{2.13}$$

where $\boldsymbol{Q}^{(k)} = (Q^{(k)}_{ij})_{ij\le N} = \mathbf{1} - \boldsymbol{P}^{(k)}$. As we show in Lemma 5.1 below, this covariance matrix itself is a projection.[1]

If $X_N$, $Y_N$ are two sequences of random variables, we write

$$X_N \simeq Y_N, \tag{2.14}$$

if there exists a constant $C > 0$, depending possibly on other parameters, but not on $N$, with

$$\mathbb{P}\big(|X_N - Y_N| > t\big) \le C\mathrm{e}^{-t^2 N/C}. \tag{2.15}$$

$X_N \simeq Y_N$ in particular implies $X_N - Y_N \to 0$ in $L_p(\mathbb{P})$ for every $p > 0$ as $N \to \infty$. By Proposition 6 of [8], we have

$$\|\boldsymbol{m}^{(k)}\| \simeq q \tag{2.16}$$

for each $k \in \mathbb{N}$.

We will also use the following lemma. We recall the definition of $\boldsymbol{h}^{(k)}$ from (2.5).

**Lemma 2.1** (Law of large numbers, Lemma 14 of [8])**.** *For any Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$ with $|f(x)| \le C(1 + |x|)$ for some constant $C < \infty$, and any $k \ge 2$, we have for $\beta > 0$, $h \neq 0$ satisfying* (1.7) *that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f\left(h^{(k)}_i\right) = \mathsf{E} f(h + \beta\sqrt{q} Z), \tag{2.17}$$

*in $L_1(\mathbb{P})$.*

## 3. Replica symmetric formula for the TAP free energy

To prove Theorem 1.3, we will use the following lemma.

**Lemma 3.1.** *Let*

$$\Delta^{(k)} = \tanh^{-1}(\boldsymbol{m}^{(k)}) - h\mathbf{1} - \frac{\beta}{\sqrt{N}} \bar{\boldsymbol{g}} \boldsymbol{m}^{(k)} + \beta^2(1-q)\boldsymbol{m}^{(k)}, \tag{3.1}$$

*and assume that $\beta > 0$, $h \neq 0$ satisfies the AT condition* (1.7)*. Then,*

$$\|\Delta^{(k)}\| \to 0 \quad \text{in } L_2(\mathbb{P}) \quad \text{as } N \to \infty \text{ followed by } k \to \infty\,. \tag{3.2}$$

*Proof.* Let us write $X \overset{N,k}{\sim} Y$ if $\|X - Y\| \to 0$ in $L_2(\mathbb{P})$ as $N \to \infty$ followed by $k \to \infty$, and $\overset{N}{\sim}$ if the norm vanishes already as $N \to \infty$. From (2.5), we have

$$\beta^{-1}\Delta^{(k)} = \sum_{s=1}^{k-2} \gamma_s \zeta^{(s)} + \sqrt{q - \Gamma^2_{k-2}}\,\zeta^{(k)} - \frac{1}{\sqrt{N}} \bar{\boldsymbol{g}} \boldsymbol{m}^{(k)} + \beta(1-q)\boldsymbol{m}^{(k)}\,. \tag{3.3}$$

[1]As we want this covariance to be a projection, we define the entries of $\boldsymbol{g}$ with unit variance, while [8] defines them with variance $1/N$. As a consequence, we have to carry along the scaling factor $N^{-1/2}$ when $\boldsymbol{g}$ is used.

We note that the second term on the right-hand side of (3.3) is $\overset{N,k}{\sim} 0$ by Lemmas 2 and 15a of [8]. Thus it holds

$$\beta^{-1}\Delta^{(k)} \overset{N,k}{\sim} \sum_{s=1}^{k-2} \gamma_s \zeta^{(s)} - \frac{1}{\sqrt{N}} \bar{\boldsymbol{g}} \boldsymbol{m}^{(k)} + \beta(1-q)\boldsymbol{m}^{(k)} \,. \tag{3.4}$$

We will now rewrite the second term of (3.4). From (2.9), we obtain

$$\bar{\boldsymbol{g}} = \bar{\boldsymbol{g}}^{(k)} + \sqrt{\tfrac{N}{2}} \sum_{s=1}^{k-1} \big(\rho^{(s)} + {\rho^{(s)}}^T\big), \tag{3.5}$$

which we use together with (2.9) in

$$\tfrac{1}{\sqrt{N}}\bar{\boldsymbol{g}}\boldsymbol{m}^{(k)} = \tfrac{1}{\sqrt{N}}\bar{\boldsymbol{g}}^{(k)}\boldsymbol{m}^{(k)} + \sum_{s=1}^{k-1} \Big[\zeta^{(s)}\langle \phi^{(s)}, \boldsymbol{m}^{(k)}\rangle + \phi^{(s)}\langle \zeta^{(s)}, \boldsymbol{m}^{(k)}\rangle - \sqrt{2}\langle \phi^{(s)}, \xi^{(s)}\rangle \phi^{(s)}\langle \phi^{(s)}, \boldsymbol{m}^{(k)}\rangle\Big] \,. \tag{3.6}$$

The expression on the right-hand side of the last display is

$$\overset{N,k}{\sim} \sum_{s=1}^{k-1}\gamma_s\zeta^{(s)} + \sum_{s=1}^{k-2}\phi^{(s)}\beta\gamma_s(1-q) + \phi^{(k-1)}\beta(1-q)\sqrt{q-\Gamma_{k-2}^2} - \sqrt{2}\sum_{s=1}^{k-1}\langle \phi^{(s)}, \xi^{(s)}\rangle\phi^{(s)}\gamma_s, \tag{3.7}$$

by Proposition 6, Lemmas 13 and 16 of [8]. As $\|\phi^{(s)}\| = 1$, the last term in (3.7) is $\overset{N}{\sim} 0$ by Lemma 11 of [8].

Replacing $\frac{1}{\sqrt{N}}\bar{\boldsymbol{g}}\boldsymbol{m}^{(k)}$ in (3.4) by (3.7), after cancellations we obtain

$$\beta^{-1}\Delta^{(k)} \overset{N,k}{\sim} -\gamma_{k-1}\zeta^{(k-1)} + \beta(1-q)\Bigg(\boldsymbol{m}^{(k)} - \sum_{s=1}^{k-2}\phi^{(s)}\gamma_s - \phi^{(k-1)}\sqrt{q-\Gamma_{k-2}^2}\Bigg). \tag{3.8}$$

By Lemmas 2 and 15a of [8], the first term on the r.h.s. of (3.8) vanishes and we obtain

$$\beta^{-1}\Delta^{(k)} \overset{N,k}{\sim} \beta(1-q)\Bigg(\boldsymbol{m}^{(k)} - \sum_{s=1}^{k-2}\phi^{(s)}\gamma_s - \phi^{(k-1)}\sqrt{q-\Gamma_{k-2}^2}\Bigg), \tag{3.9}$$

where the $\|\cdot\|$ norm of the r.h.s. is bounded by

$$\beta(1-q)\left(\Bigg\|\boldsymbol{m}^{(k)} - \sum_{s=1}^{k-2}\phi^{(s)}\gamma_s\Bigg\| + \sqrt{q-\Gamma_{k-2}^2}\Big\|\phi^{(k-1)}\Big\|\right). \tag{3.10}$$

By (2.10), we have that $\lim_{k\to\infty}\sqrt{q-\Gamma_{k-2}^2} = 0$, recalling that $\|\phi^{(k-1)}\| = 1$, the last term in the brackets on the r.h.s. of (3.10) vanishes. As for the first term in the brackets, using the fact that $\{\phi^{(s)}\}$ is an orthonormal basis, it holds

$$\Bigg\|\boldsymbol{m}^{(k)} - \sum_{s=1}^{k-2}\phi^{(s)}\gamma_s\Bigg\|^2 = \Big\|\boldsymbol{m}^{(k)}\Big\|^2 + \sum_{s=1}^{k-2}\gamma_s^2 - 2\sum_{s=1}^{k-2}\gamma_s\langle \boldsymbol{m}^{(k)}, \boldsymbol{\phi}^{(s)}\rangle, \tag{3.11}$$

By Proposition 6 of [8] together with (2.10) implies that the $\lim_{k\to\infty}\lim_{N\to\infty}$ of r.h.s. of the latter is equal to 0. $\square$

We are now ready to prove the convergence of the TAP functional to the replica-symmetric free energy:

*Proof of Theorem 1.3.* To see how this goes, we first reformulate (1.8) with the help of (1.9) with the right scaling,

$$\begin{aligned} N^{-1}\mathrm{TAP}_N(\boldsymbol{m}^{(k)}) &= \tfrac{\beta}{2}N^{-3/2}\sum_{i\neq j}^{N}\bar{g}_{ij}m_i^{(k)}m_j^{(k)} + hN^{-1}\sum_{i=1}^{N}m_i^{(k)} - N^{-1}\sum_{i=1}^{N}\tanh^{-1}(m_i^{(k)})m_i^{(k)} \\ &\quad + N^{-1}\sum_{i=1}^{N}\log\cosh\tanh^{-1}(m_i^{(k)}) + \frac{\beta^2}{4}\left(1-\frac{1}{N}\sum_{i=1}^{N}{m_i^{(k)}}^2\right)^2. \end{aligned} \tag{3.12}$$

By Lemma 2.1, the terms in the second line of the latter converge in $L_1(\mathbb{P})$ to the r.h.s. of (1.19) as $N\to\infty$.

It remains to show that the sum of the first three terms on the r.h.s. of (3.12) converges to 0 in $L_1(\mathbb{P})$ as $N\to\infty$, followed by $k\to\infty$. By Lemma 2.1, first note that the limit in $L_1(\mathbb{P})$ of the second and third term on the right-hand side of (3.12) is

$$\begin{aligned} \lim_{N\to\infty} N^{-1}\sum_{i=1}^{N}\left(h-\tanh^{-1}(m_i^{(k)})\right)m_i^{(k)} &= -\mathsf{E}\left(\beta\sqrt{q}Z\tanh\left(h+\beta\sqrt{q}Z\right)\right) \\ &= -\beta^2 q(1-q), \end{aligned} \tag{3.13}$$

the last line by combining a simple integration by parts with (1.6). It only remains to prove that the first term on the r.h.s. (3.12) tends to $\beta^2 q(1-q)$. By Lemma 3.1, it holds

$$\frac{\beta}{\sqrt{N}}\sum_{j\colon j\neq i}\bar{g}_{ij}m_j^{(k)} = -h+\beta^2(1-q)m_i^{(k)}+\tanh^{-1}(m_i^{(k)})-\Delta_i^{(k)}-\frac{\beta}{\sqrt{N}}\bar{g}_{ii}m_i^{(k)}, \tag{3.14}$$

Multiplying the latter by $\frac{m_i^{(k)}}{2N}$ and taking the sum over $i$ yield

$$\begin{aligned} \tfrac{1}{2}\beta N^{-3/2}\sum_{i\neq j}^{N}\bar{g}_{ij}m_i^{(k)}m_j^{(k)} &= -\tfrac{1}{2}hN^{-1}\sum_{i=1}^{N}m_i^{(k)} + \tfrac{1}{2}\beta^2(1-q)N^{-1}\sum_{i=1}^{N}{m_i^{(k)}}^2 \\ &\quad + \tfrac{1}{2}N^{-1}\sum_{i=1}^{N}m_i^{(k)}\tanh^{-1}(m_i^{(k)}) - \tfrac{1}{2}N^{-1}\sum_{i=1}^{N}m_i^{(k)}\Delta_i^{(k)} - \frac{\beta}{2N\sqrt{N}}\sum_{i=1}^{N}\bar{g}_{ii}{m_i^{(k)}}^2. \end{aligned} \tag{3.15}$$

The last term on the r.h.s. of (3.15) tends to 0 in $L_2(\mathbb{P})$ as $N\to\infty$ as

$$\mathbb{E}\frac{1}{N}\sum_{i=1}^{N}\left|\frac{\beta}{\sqrt{N}}\bar{g}_{ii}m_i^{(k)}\right|^2 \leq \frac{\beta}{N^2}\sum_{i=1}^{N}\mathbb{E}\bar{g}_{ii}^2 = \frac{\beta}{N}. \tag{3.16}$$

Combining Cauchy-Schwarz with Lemma 3.1 and (2.16), we have that the second last term on the r.h.s. of (3.15) tends to 0 in $L_2(\mathbb{P})$ as $N\to\infty$ followed by $k\to\infty$. The sum of the remaining terms on the r.h.s. of (3.15) converges, as $N\to\infty$ in $L_1(\mathbb{P})$ by

Lemma 2.1 to

$$-\tfrac{1}{2}h\mathsf{E}\left(\tanh\left(h+\beta\sqrt{q}Z\right)\right)+\tfrac{1}{2}\beta^2(1-q)\mathsf{E}\left(\tanh^2\left(h+\beta\sqrt{q}Z\right)\right)$$
$$+\tfrac{1}{2}\mathsf{E}\left(\tanh\left(h+\beta\sqrt{q}Z\right)\left(h+\beta\sqrt{q}Z\right)\right)=\beta^2q(1-q), \quad (3.17)$$

again using (3.13) and (1.6). All in all, we obtain that

$$\lim_{N\to\infty,\, k\to\infty}\tfrac{1}{2}\beta N^{-\frac{3}{2}}\sum_{i\neq j}^{N}\bar{g}_{ij}m_i^{(k)}m_j^{(k)}=\beta^2q(1-q), \text{in } L_1(\mathbb{P}). \quad (3.18)$$

We proved that the first term on the r.h.s. of (3.12) tends to $\beta^2q(1-q)$ and the assertion of Theorem 1.3 follows. □

## 4. Gaussian orthogonal ensemble

As a tool to study the Hessian of the TAP free energy functional, we record some known facts about the Gaussian orthogonal ensemble (GOE). A GOE with variance $\sigma^2>0$ is a real symmetric random matrix $\boldsymbol{X}$ with centered Gaussian entries of variance $\sigma^2$ off the diagonal, variance $2\sigma^2$ on the diagonal, and the entries $(X_{ij})_{1\le i\le j\le N}$ being independent. The matrix $(\beta N^{-1/2}\bar{g}_{ij})_{i,j=1,\ldots,N}$ is a GOE with variance $\beta^2/N$. Thus, by Wigner's Theorem (see e. g. Theorem 2.1.1 in [3]), its empirical spectral distribution converges weakly in probability to the semicircle law $\mu_\beta$ which is defined by its density

$$\frac{\mathrm{d}\mu_\beta(x)}{\mathrm{d}x}=1_{[-2\beta,2\beta]}(x)\sqrt{4\beta^2-x^2}\,. \quad (4.1)$$

Also, the largest eigenvalue $\lambda_1(\beta N^{-1/2}\bar{\boldsymbol{g}})$ converges a. s. to $2\beta$ (see e. g. Theorem 1.13 of [34]).

For each real symmetric matrix $\boldsymbol{M}$ of size $n$, we denote the enumeration of its eigenvalues in non-increasing order by $\lambda_1(\boldsymbol{M})\ge\ldots\ge\lambda_n(\boldsymbol{M})$, and its empirical spectral distribution by

$$\mu_{\boldsymbol{M}}:=\frac{1}{N}\sum_{i=1}^{n}\delta_{\lambda_i(\boldsymbol{M})}. \quad (4.2)$$

We recall that the Frobenius norm of a matrix $\boldsymbol{M}$ of size $n$ is defined by $\|\boldsymbol{M}\|_{\mathrm{F}}=(\sum_{i,j=1}^{N}|M_{ij}|^2)^{1/2}$. The following standard result, for which we refer to Exercises 2.4.3 and 2.4.4 of [33], states that the limiting empirical spectral distributions of random matrices are invariant under additive perturbations in the prelimiting sequence that have either small rank or small Frobenius norm.

**Lemma 4.1.** *Let $\boldsymbol{M}_n$ and $\boldsymbol{N}_n$ be random Hermitian matrices of size $n$ such that the empirical spectral distribution of $\boldsymbol{M}_n$ converges weakly a. s. to a probability measure $\mu$. Suppose that at least one of the following conditions holds true:*

*(i)* $n^{-1}\|\boldsymbol{N}_n\|_{\mathrm{F}}^2\to 0$ *a. s.* ,
*(ii)* $n^{-1}\mathrm{rank}(\boldsymbol{N}_n)\to 0$ *a. s.* .

*Then the empirical spectral distribution of $\boldsymbol{M}_n+\boldsymbol{N}_n$ converges to $\mu$ weakly a. s.* .

Similarly, for the largest eigenvalue we have:

**Lemma 4.2.** *Let $\boldsymbol{M}_n$ and $\boldsymbol{N}_n$ be random Hermitian matrices of size $n$ such that the largest eigenvalue $\lambda_1(\boldsymbol{M}_n)$ of $\boldsymbol{M}_n$ converges a. s. to a limit $\lambda_1$ as $n \to \infty$. Suppose that $\|\boldsymbol{N}_n\|_{\mathrm{F}}^2 \to 0$ a. s. Then also the largest eigenvalue $\lambda_1(\boldsymbol{M}_n + \boldsymbol{N}_n)$ of $\boldsymbol{M}_n + \boldsymbol{N}_n$ converges to the same limit $\hat{\lambda}_1$ almost surely.*

*Proof.* This follows from the Hoffman-Wielandt inequality (below Lemma 2.4.3 of [33]). □

4.1. **Free convolution.** First we state a definition of the free convolution (see [6, 37, 23]). The Stieltjes transform of a probability measure $\mu$ on $\mathbb{R}$ is defined by

$$g_\mu(z) = \int \frac{\mathrm{d}\mu(x)}{z - x} \tag{4.3}$$

which is analytic in $\mathbb{C} \setminus \operatorname{supp} \mu$. It can be shown that there exists a domain $D$ on which $g_\mu$ is univalent. Denoting by $K_\mu$ the inverse function of $g_\mu$ defined on $g_\mu(D)$, the R-transform of $\mu$ is defined on $g_\mu(D)$ by

$$R_\mu(z) = K_\mu(z) - \frac{1}{z}\,. \tag{4.4}$$

Free probability theory shows that for probability measures $\mu, \lambda$ on $\mathbb{R}$, there exists a unique probability measure $\kappa$ with

$$R_\kappa = R_\lambda + R_\mu \tag{4.5}$$

on a domain on which these three functionals are defined. The measure $\kappa$ is denoted by $\lambda \boxplus \mu$ and called the free (additive) convolution of $\lambda$ and $\mu$.

The following result ensures that limiting spectral distribution of a sum of a GOE and a deterministic matrix whose spectral distribution weakly converges is given by a free additive convolution with the semicircle law. The support of this free convolution is analyzed in Lemma 6.1 below.

**Lemma 4.3.** *For $n \in \mathbb{N}$, let $\boldsymbol{X}_n$ be a GOE with unit variance, and let $\boldsymbol{A}_n$ be a deterministic real and symmetric matrix, each of size $n$, such that the empirical spectral distribution $\mu_{\boldsymbol{A}_n}$ converges weakly to some probability measure $\nu$ on $\mathbb{R}$ as $n \to \infty$. Then, for each $\sigma > 0$, the empirical spectral distribution of $\sigma n^{-1/2}\boldsymbol{X}_n + \boldsymbol{A}_n$ converges weakly almost surely to $\mu_\sigma \boxplus \nu$.*

*Proof.* This is a standard result from free probability theory, see for example Theorem 5.4.5 in [3]. Also, Pastur [22] (p.12) gives a functional equation solved by the Stieltjes transform of the limiting spectral distribution of $\sigma n^{-1/2}\boldsymbol{X}_n + \boldsymbol{A}_n$. The Stieltjes transform of the limiting distribution solves a functional equation, which has a unique solution [26], (p.69). We conclude with the fact that the Stieltjes transform of $\mu_\sigma \boxplus \nu$ solves the same functional equation (see e.g. Proposition 2.1 in [11]). □

We will also use the following version of a result of Capitaine et al. [11] for the largest eigenvalue. For $\sigma > 0$ and a probability measure $\nu$ on $\mathbb{R}$, let

$$H_{\sigma,\nu}(z) := z + \sigma^2 g_\nu(z) \tag{4.6}$$

and

$$\mathcal{O}_{\sigma,\nu} := \{u \in \mathbb{R} \setminus \operatorname{supp} \nu \ : H'_{\sigma,\nu}(u) > 0\} \tag{4.7}$$

where $g_\nu$ denotes the Stieltjes transform defined as in (4.3).

**Lemma 4.4** (cf. [11], Theorem 8.1)**.** *Let $\sigma > 0$, let $\boldsymbol{X}_N$ be a GOE with unit variance, and let $\boldsymbol{A}_N$ be a deterministic real and symmetric matrix. Assume that the empirical spectral distribution $\mu_{\boldsymbol{A}_N}$ converges weakly to a probability measure $\nu$ on $\mathbb{R}$ as $N \to \infty$, and that there exists $d \in \mathbb{R}$ with $\nu(d, \infty) = 0$. Also, suppose that there exist an integer $r \geq 2$ and $\theta \in \mathbb{R} \setminus \operatorname{supp} \nu$ with $\lim_{N\to\infty} \lambda_1(\boldsymbol{A}_N) = \theta$ and*

$$\max_{j=r,\ldots,N} \mathrm{d}(\lambda_j(\boldsymbol{A}_N), \operatorname{supp}(\nu)) \xrightarrow{N\to\infty} 0. \tag{4.8}$$

*Then the following holds:*

*(i) If $\theta \in \mathcal{O}_{\sigma,\nu}$, then $\lim_{N\to\infty} \lambda_1(\sigma N^{-1/2}\boldsymbol{X}_N + \boldsymbol{A}_N) = H_{\sigma,\nu}(\theta)$ almost surely.*

*(ii) If $\theta \in \mathbb{R} \setminus \mathcal{O}_{\sigma,\nu}$, then $\lim_{N\to\infty} \lambda_1(\sigma N^{-1/2}\boldsymbol{X}_N + \boldsymbol{A}_N) = \max \operatorname{supp} \nu$ almost surely.*

*Proof.* We abbreviate $\boldsymbol{M}_N := \sigma N^{-1/2}\boldsymbol{X}_N + \boldsymbol{A}_N$. Consider an orthogonal diagonalization $\boldsymbol{A}_N = \boldsymbol{O}_N^T \boldsymbol{D}_N \boldsymbol{O}_N$ of $\boldsymbol{A}_N$. As $\boldsymbol{O}_N \boldsymbol{X}_N \boldsymbol{O}_N^T$ is again distributed as a GOE, and as $\boldsymbol{M}_N$ has the same eigenvalues as $\boldsymbol{O}^T \boldsymbol{M}_N \boldsymbol{O}_N$, we henceforth assume w. l. o. g. that $\boldsymbol{A}_N$ is diagonal.

We infer the assertions from Theorem 8.1 of [11] in the case that $\nu$ has compact support. First, the proof of Theorem 8.1 of [11] passes through for GOE (in place of GUE) when Theorem 5.1 of [11] is replaced with Theorem 4.2 of [16]. We write $\gamma_1 := \theta$. We assume w. l. o. g. that $r$ is the minimal integer satisfying assumption (4.8). For any subsequence of $N$ tending to infinity, we find a subsubsequence $(N_i)$ tending to infinity along which $\lambda_j(\boldsymbol{A}_{N_i})$ converges to some $\gamma_j \in (d, \theta]$ for all $j = 2, \ldots, r$, using compactness of the interval $[d, \theta]$ and minimality of $r$. Hence, there exists a diagonal matrix $\widetilde{\boldsymbol{A}}_{N_i}$ with eigenvalues $\lambda_j(\widetilde{\boldsymbol{A}}_{N_i}) = \gamma_j$ whose difference to $\boldsymbol{A}_{N_i}$ vanishes in the Frobenius norm

$$\|\widetilde{\boldsymbol{A}}_{N_i} - \boldsymbol{A}_{N_i}\|_{\mathrm{F}} \xrightarrow{i\to\infty} 0. \tag{4.9}$$

From (4.9), it follows that $\boldsymbol{A}_{N_i}$ can be replaced with $\widetilde{\boldsymbol{A}}_{N_i}$ in the definition of $\boldsymbol{M}_{N_i}$ without changing the limiting largest eigenvalue by Lemma 4.2.

We now assume that $\theta \in \mathcal{O}_{\sigma,\nu}$ and show assertion (i). In this case, $\boldsymbol{A}_{N_i}$ satisfies the assumptions of Theorem 8.1 1) of [11], which yields

$$\lim_{i\to\infty} \lambda_1(\boldsymbol{M}_{N_i}) = H_{\beta,\nu}(\theta) \quad \text{a. s.} \tag{4.10}$$

As the limit in (4.10) does not depend on the choice of the subsequence of $N$, it also holds for the original sequence along which $N \to \infty$.

It remains to consider the case of the more general $\nu$ in the assertion. For this, we use truncation arguments for matching upper and lower bounds.

*Lower bound.* For $m \in \mathbb{R}_+$, we consider

$$V_m := \{i = 1, \ldots, N : \ A_{ii} \geq -m\} \tag{4.11}$$

which records the diagonal entries of $\boldsymbol{A}_N$ that have a value at least $-m$. The number of those diagonal entries will be denoted by $N_m = \#V_m$, and we set $r_{m,N} := \sqrt{N_m/N}$. Now,

$$\begin{aligned}
\lambda_1(\boldsymbol{M}_N) &= \sup\left\{\boldsymbol{v}^T \boldsymbol{M}_N \boldsymbol{v} : \boldsymbol{v} \in \mathbb{R}^N, \|\boldsymbol{v}\|_2 = 1\right\} \\
&\geq \sup\left\{\boldsymbol{v}^T \boldsymbol{M}_N \boldsymbol{v} : \boldsymbol{v} \in \mathbb{R}^N, \|\boldsymbol{v}\|_2 = 1, \max_{i\notin V_m} |v_i| = 0\right\} \\
&= r_{m,N} \sup\left\{\boldsymbol{v}^T \boldsymbol{M}_N^{(m)} \boldsymbol{v} : \boldsymbol{v} \in \mathbb{R}^{N_m}, \|\boldsymbol{v}\|_2 = 1\right\} = r_{m,N} \lambda_1\left(\boldsymbol{M}_N^{(m)}\right)
\end{aligned} \tag{4.12}$$

where

$$\boldsymbol{M}_N^{(m)} = \sigma N_m^{-1/2}\boldsymbol{X}_N^{(m)} + \boldsymbol{A}_N^{(m)}, \quad \boldsymbol{X}_N^{(m)} = (X_{f(i),f(j)})_{i,j\le N_m}, \quad \boldsymbol{A}_N^{(m)} = {r_{m,N}}^{-1}(A_{f(i),f(j)})_{i,j\le N_m} \tag{4.13}$$

and $f(i)$ denotes the $i$-th largest integer in $V_m$. Note that $\boldsymbol{X}_N^{(m)}$ is again a GOE of size $N_m$, that $\lim_{N\to\infty} r_{m,N}^2 = \nu([-m,d]) =: r_m^2$ for all but countably many $m$, and that $\mu_{\boldsymbol{A}_N^{(m)}}$ weakly converges to the probability measure $\nu_m$ as $N\to\infty$, where $\nu_m$ is defined as the image measure of $\nu(\cdot\cap[-m,d])/\nu([-m,d])$ under the dilation $t\mapsto r_m^{-1}t$. Also $r_m\to 1$ as $m\to\infty$ by definition of $r_m$. For $m\ge -2\theta$ and all $N$, we have $r_{m,N}\lambda_1(\boldsymbol{A}_N^{(m)}) = \theta$, and hence $\lambda_1(\boldsymbol{A}_N^{(m)}) \to r_m^{-1}\theta$ as $N\to\infty$. Moreover, we note that $\theta\notin \operatorname{supp}\nu_m$ for sufficiently large $m$, and from (4.6), we obtain $\lim_{m\to\infty} H_{\sigma,\nu_m}(\theta) = H_{\sigma,\nu}(\theta)$. By differentiating (4.6) and using the definition (4.3) of the Stieltjes transform, we also obtain

$$H'_{\beta,\nu_m}(\theta) = 1-\beta^2\int\frac{\nu_m(\mathrm{d}x)}{(\theta-x)^2} = 1-\frac{\beta^2}{\nu[-m,d]}\int_{[-m,d]}\frac{\nu(\mathrm{d}x)}{(\theta-r_m^{-1}x)^2}, \tag{4.14}$$

which converges to $H'_{\beta,\nu}(\theta)$ as $m\to\infty$. Hence, $H'_{\sigma,\nu_m}(\theta)>0$ for sufficiently large $m$. As $\nu_m$ is compactly supported, the first part of the proof yields $\lim_{N\to\infty}\lambda_1(\boldsymbol{M}_N^{(m)}) = r_m H_{\sigma,\nu_m}(\theta)$ a. s. Using (4.12) and taking $m\to\infty$ yields $\liminf_{N\to\infty}\lambda_1(\boldsymbol{M}_N)\ge H_{\sigma,\nu}(\theta)$ almost surely.

*Upper bound.* We use the truncation $\widehat{\boldsymbol{A}}_N^{(m)} := \operatorname{diag}(A_{ii}\vee(-m))_{i=1,\dots,N}$, and we set $\widehat{\boldsymbol{M}}_N^{(m)} := \sigma N^{-1/2}\boldsymbol{X}_N + \widehat{\boldsymbol{A}}_N^{(m)}$. In place of (4.12), we then have

$$\lambda_1(\boldsymbol{M}_N) = \sup\left\{\boldsymbol{v}^T\boldsymbol{M}_N\boldsymbol{v} : \boldsymbol{v}\in\mathbb{R}^N, \|\boldsymbol{v}\|_2 = 1\right\} \le \lambda_1(\widehat{\boldsymbol{M}}_N^{(m)}) \tag{4.15}$$

as

$$\boldsymbol{v}^T\boldsymbol{A}_N\boldsymbol{v} = \sum_{i=1}^N v_i^2 A_{ii} \le \sum_{i=1}^N v_i^2\widehat{A}_{ii}^{(m)} = \boldsymbol{v}^T\widehat{\boldsymbol{A}}_N^{(m)}\boldsymbol{v}. \tag{4.16}$$

The empirical spectral distribution $\mu_{\widehat{\boldsymbol{A}}_N^{(m)}}$ weakly converges to $\nu(\cdot\cap[-m,d])+\nu(-\infty,-m)\delta_{-m}$, and we conclude in the same way as for the lower bound.

To show assertion (ii), we assume that $\theta\in\mathbb{R}\setminus\mathcal{O}_{\sigma,\nu}$. Then it follows that $\theta\in\overline{\{u : H'_{\sigma,\nu}(u)>0\}}$ by continuity (see also [11], p. 1754) and assertion (ii) follows from Theorem 8.1 2a) of [11] by the above truncation argument, where it suffices to consider the upper bound. □

## 5. Conditional Hessian

To analyze the spectral behavior of the Hessian $\boldsymbol{H}^{(k)}$ from (1.21), it is useful to condition on the $\sigma$-algebra $\mathcal{G}_{k-1}$ with respect to which the magnetization $\boldsymbol{m}^{(k)}$ is measurable. Under this conditioning, $\boldsymbol{g}^{(k)}$ remains centered Gaussian with covariances given by (2.13). In the present section, we show that up to a negligible additive error (as for Lemma 4.1), $\bar{\boldsymbol{g}}^{(k)}$ can be considered as a GOE also under the conditioning on $\mathcal{G}_{k-1}$. Thus we obtain a representation of $\boldsymbol{H}^{(k)}$ as the sum of a GOE and independent $\mathcal{G}_{k-1}$-measurable terms.

First we give some properties of the covariance matrix $\boldsymbol{V}^{(k)} = (V_{ij,st}^{(k)})_{i,j,s,t\le N}$ of $\boldsymbol{g}^{(k)}$ under $\mathbb{P}(\cdot\mid\mathcal{G}_{k-1})$ which follow from its definition (2.13) in terms of the projection $\boldsymbol{Q}^{(k)}$. In the following, we will denote by $\mathbb{P}_{k-1} := \mathbb{P}(\cdot\mid\mathcal{G}_{k-1})$ with associated expectation $\mathbb{E}_{k-1}$ the conditional probability given $\mathcal{G}_{k-1}$.

**Lemma 5.1.** *The matrix $\boldsymbol{V}^{(k)}$ is a projection, that is, $\boldsymbol{V}^{(k)} = \boldsymbol{V}^{(k)^2}$. Furthermore, $\boldsymbol{V}^{(k)} = \mathbf{1}+\boldsymbol{J}$ for a matrix $\boldsymbol{J}$, where $\boldsymbol{J}$ has eigenvalue $-1$ with multiplicity $N^2-(N-k+1)^2$, and all other eigenvalues are zero.*

*Proof.* By (2.11) and as $\boldsymbol{Q}^{(k-1)}$ is a projection,

$$V^{(k)}_{ij,st} = Q^{(k-1)}_{is} Q^{(k-1)}_{jt} = \left(\sum_{u=1}^{N} Q^{(k-1)}_{iu} Q^{(k-1)}_{us}\right)\left(\sum_{v=1}^{N} Q^{(k-1)}_{jv} Q^{(k-1)}_{vt}\right) = \sum_{u,v=1}^{N} V^{(k)}_{ij,uv} V^{(k)}_{uv,st}, \tag{5.1}$$

which shows that $\boldsymbol{V}^{(k)}$ is a projection.

To show the assertion on the eigenvalues of $\boldsymbol{J}$, we first note that

$$\boldsymbol{Q}^{(k-1)} = \mathbf{1} - \boldsymbol{P}^{(k-1)} = \mathbf{1} - \frac{1}{N}\sum_{s=1}^{k-1} \phi^{(s)} \phi^{(s)^T} = \mathbf{1} - \boldsymbol{O}\left(\sum_{s=1}^{k-1} \boldsymbol{D}^{(s)}\right)\boldsymbol{O}^T, \tag{5.2}$$

where $\boldsymbol{O}$ is an orthogonal matrix and $\boldsymbol{D}^{(s)}$ are diagonal matrices with one entry equal to 1, and the other entries equal to 0. The last equality is due to the fact that $\boldsymbol{P}^{(k-1)}$ is a sum of projectors of rank 1 to orthogonal subspaces: thus, these projectors are orthogonally diagonalisable in the same basis. Let

$$\boldsymbol{D} = \mathbf{1} - \sum_{s=1}^{k-1} \boldsymbol{D}^{(s)}, \tag{5.3}$$

one readily checks that $\boldsymbol{D}$ has $k-1$ entries that are equal to 0 and the rest equal to 1. Defining $\boldsymbol{J}$ by $\boldsymbol{V}^{(k)} = \mathbf{1} + \boldsymbol{J}$ and using the definition (2.13) of $\boldsymbol{V}^{(k)}$, we obtain

$$J_{ij,st} = Q^{(k-1)}_{is} Q^{(k-1)}_{jt} - \delta_{ij,st} = \sum_{u,v=1}^{N} O_{ui} D_{uu} O_{us} O_{vj} D_{vv} O_{vt} - \delta_{ij,st}. \tag{5.4}$$

Next we define $\widetilde{\boldsymbol{O}}$ and $\widetilde{\boldsymbol{D}}$ by $\widetilde{O}_{ij,st} = O_{is}O_{jt}$ and $\widetilde{D}_{ij,st} = D_{is}D_{jt}$. Then $\widetilde{\boldsymbol{O}}$ is orthogonal as

$$(\widetilde{\boldsymbol{O}}^T \widetilde{\boldsymbol{O}})_{ij,st} = \sum_{u,v=1}^{N} O_{i,u} O_{j,v} O_{u,s} O_{v,t} = (\boldsymbol{O}^T\boldsymbol{O})_{is} (\boldsymbol{O}^T\boldsymbol{O})_{jt} = \delta_{is}\delta_{jt}. \tag{5.5}$$

Hence, we get from (5.4) that $\boldsymbol{J} = \widetilde{\boldsymbol{O}}^T(\widetilde{\boldsymbol{D}} - \mathbf{1})\widetilde{\boldsymbol{O}}$. As the diagonal matrix $(\widetilde{\boldsymbol{D}} - \mathbf{1})$ has $(N-k+1)^2$ entries equal to zero, the other entries being $-1$, the assertion follows. □

As a consequence, we can approximate $\bar{\boldsymbol{g}}^{(k)}$ by a GOE:

**Lemma 5.2.** *Under $\mathbb{P}(\cdot \mid \mathcal{G}_{k-1})$, there exists a GOE $\mathbf{X}$ such that $\bar{\boldsymbol{g}}^{(k)} = \boldsymbol{X} + \boldsymbol{Y}$ and $N^{-1/2}\|\boldsymbol{Y}\|_{\mathrm{F}}$ is tight in $N$.*

*Proof.* From (2.13) and as $\boldsymbol{V}^{(k)} = \boldsymbol{V}^{(k)^2}$ by Lemma 5.1, there exists a vector $\boldsymbol{Z} = (Z_{ij})_{ij\le N}$ of length $N^2$ whose entries are iid standard Gaussians, such that $g^{(k)}_{ij} = (\boldsymbol{V}^{(k)}\boldsymbol{Z})_{ij}$ for all $i,j \le N$. Using again Lemma 5.1, we diagonalize $\boldsymbol{V}^{(k)} - \mathbf{1} = \boldsymbol{O}^T\boldsymbol{D}\boldsymbol{O}$, where $\boldsymbol{O}$ is a $\mathcal{G}_{k-1}$-measurable orthogonal matrix, and $\boldsymbol{D}$ is a deterministic diagonal matrix with $N^2-(N-k+1)^2$ entries equal to $-1$ and the rest to 0. Then we write

$$g_{ij}^{(k)} = \big((\boldsymbol{V}^{(k)}-\mathbf{1})\boldsymbol{Z}\big)_{ij} + Z_{ij} = \big(\boldsymbol{O}^T\boldsymbol{D}\boldsymbol{O}\boldsymbol{Z}\big)_{ij} + Z_{ij}. \tag{5.6}$$

We have $\bar{\boldsymbol{g}} = \boldsymbol{Y}+\boldsymbol{X}$, where

$$Y_{ij} := \frac{1}{\sqrt{2}}[\big(\boldsymbol{O}^T\boldsymbol{D}\boldsymbol{O}\boldsymbol{Z}\big)_{ij} + \big(\boldsymbol{O}^T\boldsymbol{D}\boldsymbol{O}\boldsymbol{Z}\big)_{ji}], \quad X_{ij} := \frac{1}{\sqrt{2}}[Z_{ij}+Z_{ji}], \tag{5.7}$$

where $\boldsymbol{X},\boldsymbol{Y}$ are $N\times N$ matrices. It remains to prove that $N^{-1/2}\|\boldsymbol{Y}\|_{\mathrm{F}}$ is tight in $N$. By a simple convexity argument,

$$\|\boldsymbol{Y}\|_{\mathrm{F}}^2 = \sum_{i,j=1}^N Y_{ij}^2 \le \sqrt{2}\sum_{i,j=1}^N \left[\Big(\big(\boldsymbol{O}^T\boldsymbol{D}\boldsymbol{O}\boldsymbol{Z}\big)_{ij}\Big)^2 + \Big(\big(\boldsymbol{O}^T\boldsymbol{D}\boldsymbol{O}\boldsymbol{Z}\big)_{ji}\Big)^2\right]. \tag{5.8}$$

By symmetry, it remains to consider

$$\sum_{i,j=1}^N \Big(\big(\boldsymbol{O}^T\boldsymbol{D}\boldsymbol{O}\boldsymbol{Z}\big)_{ij}\Big)^2 = \|\boldsymbol{O}^T\boldsymbol{D}\boldsymbol{O}\boldsymbol{Z}\|_{\ell_2(\mathbb{R}^{N^2})}^2 \tag{5.9}$$

and to show that this expression, when multiplied by $N^{-1/2}$, is tight in $N$. As the $\ell_2$-norm is invariant under orthogonal transformations, and as $\boldsymbol{O}\boldsymbol{Z}$ is again standard Gaussian distributed, we have

$$\|\boldsymbol{O}^T\boldsymbol{D}\boldsymbol{O}\boldsymbol{Z}\|_{\ell_2(\mathbb{R}^{N^2})} = \|\boldsymbol{D}\boldsymbol{O}\boldsymbol{Z}\|_{\ell_2(\mathbb{R}^{N^2})} \overset{d}{=} \|\boldsymbol{D}\boldsymbol{Z}\|_{\ell_2(\mathbb{R}^{N^2})}. \tag{5.10}$$

Note that $N^2-(N-k+1)^2$ many entries of the vector $\boldsymbol{D}\boldsymbol{Z}$ are $\mathcal{N}(0,1)$-distributed, the other entries being 0. Therefore, by the law of large numbers, $N^{-1}\|\boldsymbol{D}\boldsymbol{Z}\|_{\ell_2(\mathbb{R}^{N^2})}^2$ is tight in $N$, which yields the assertion. □

We consider the Hessian $\boldsymbol{H}^{(k)}$ from (1.21) which reads

$$H_{ij}^{(k)} = \frac{\beta}{\sqrt{N}}\bar{g}_{ij} + \frac{2\beta^2}{N}m_i^{(k)}m_j^{(k)}, \quad i,j=1,\dots,N, i\neq j$$

$$H_{ii}^{(k)} = -\beta^2\left(1-\frac{1}{N}\sum_{p=1}^N m_p^{(k)2}\right) - \frac{1}{1-m_i^{(k)2}} + \frac{2\beta^2}{N}m_i^{(k)2}. \tag{5.11}$$

Now we obtain the following approximation under $\mathbb{P}$:

**Lemma 5.3.** *Let the diagonal matrix $\boldsymbol{A}^{(k)}$ be defined by*

$$A_{ii}^{(k)} = -\frac{1}{1-m_i^{(k)2}}, i=1,\dots,N, \quad A_{ij}^{(k)} = 0 \textit{ for } i\neq j, \tag{5.12}$$

*and let*

$$\boldsymbol{B}^{(k)} = 2\beta^2\boldsymbol{m}^{(k)}\otimes\boldsymbol{m}^{(k)} + \beta\sum_{s=1}^{k-1}\big(\zeta^{(s)}\otimes\phi^{(s)} + \phi^{(s)}\otimes\zeta^{(s)}\big). \tag{5.13}$$

*Then, with $\boldsymbol{X}$ from Lemma 5.2,*

$$\boldsymbol{H}^{(k)} = \frac{\beta}{\sqrt{N}}\boldsymbol{X} + \boldsymbol{A}^{(k)} + \boldsymbol{B}^{(k)} - \beta^2(1-q)\mathbf{1} + \boldsymbol{R} - \epsilon\mathbf{1} \tag{5.14}$$

*where, in $\mathbb{P}(\cdot \mid \mathcal{G}_{k-1})$-probability, $N^{-1}\|\boldsymbol{R}\|_{\mathrm{F}} \to 0$ and $\epsilon \to 0$, as $N \to \infty$.*

*Proof.* We set $\epsilon = \beta^2(q - \|\boldsymbol{m}^{(k)}\|^2)$, then $\epsilon \to 0$ in probability as $N \to \infty$ by (2.16). Using the definitions (5.11), (2.8) and (2.9), we can then set

$$\boldsymbol{R} = -\sqrt{2}\sum_{s=1}^{k-1}\langle \phi^{(s)}, \xi^{(s)}\rangle(\phi^{(s)} \otimes \phi^{(s)}) + \beta N^{-1/2}\boldsymbol{Y} \tag{5.15}$$

with $\boldsymbol{Y}$ from Lemma 5.2, so that $N^{-1}\|\boldsymbol{R}\|_F$ converges to zero in probability: for the first term on the r.h.s. of (5.15), we note that $\|\phi^{(s)} \otimes \phi^{(s)}\|_{\mathrm{F}}^2 = \|\phi^{(s)}\|^2 = 1$, hence it suffices to show for each $s$ that $\langle \phi^{(s)}, \zeta^{(s)}\rangle \to 0$ in $\mathbb{P}(\cdot \mid \mathcal{G}_{k-1})$-probability as $N \to \infty$. This, however, follows from Lemma 11 of [8] which states that $\langle \phi^{(s)}, \xi^{(s)}\rangle$ is a centered Gaussian with variance $1/N$ under $\mathbb{P}$, hence it converges to 0 $\mathbb{P}$-a. s with Borel-Cantelli. □

## 6. Proof for weak limit of spectral distribution

The proof of Theorem 1.4 comes in two parts: first we show, using Bolthausen's algorithm, that $\boldsymbol{H}^{(k)}$ can be considered asymptotically as $N \to \infty$ followed by $k \to \infty$ as the sum of a GOE with variance $\beta/N$, a deterministic diagonal matrix $-\beta^2(1-q)\mathbf{1}$, and an independent diagonal matrix with independent entries having distribution

$$\nu := \mathcal{L}\left(-\frac{1}{1 - \tanh^2(h + \beta\sqrt{q}Z)}\right), \tag{6.1}$$

$Z$ being a standard Gaussian. The limiting spectrum of such a sum can be characterized as a free convolution. We also set $\hat{\nu} := \nu(\cdot + \beta^2(1-q))$, then $\hat{\nu}$ is the image measure of $\nu$ under the shift $t \mapsto t - \beta^2(1-q)$.

*Proof of Theorem 1.4.* We can rewrite $\boldsymbol{B}^{(k)}$ as

$$\boldsymbol{B}^{(k)} = 2\beta^2\boldsymbol{m}^{(k)} \otimes \boldsymbol{m}^{(k)} + \tfrac{1}{2}\beta\sum_{s=1}^{k-1}\left[\left(\zeta^{(s)} + \phi^{(s)}\right) \otimes \left(\zeta^{(s)} + \phi^{(s)}\right) - \left(\zeta^{(s)} - \phi^{(s)}\right) \otimes \left(\zeta^{(s)} - \phi^{(s)}\right)\right] \tag{6.2}$$

which is a sum of $2k-1$ matrices of rank 1. Hence, by Lemma 4.1 (and induction over $k$), $\boldsymbol{B}^{(k)}$ has no influence on the limiting spectral distribution of $\boldsymbol{H}^{(k)}$ as $N \to \infty$. Thus, the empirical spectral distribution of $\boldsymbol{M} := \beta N^{-1/2}\boldsymbol{X} + \boldsymbol{A}^{(k)} - \beta^2(1-q)\mathbf{1}$ converges by Lemmas 4.1 and 5.3 and Slutzky's lemma to the same weak limit as $\mu_{\boldsymbol{H}^{(k)}}$ a. s. as $N \to \infty$ followed by $k \to \infty$.

By Lemma 2.1, we have

$$\int \mu_{\boldsymbol{A}^{(k)}}(\mathrm{d}x)f(x) \longrightarrow \int \nu(\mathrm{d}x)f(x) \tag{6.3}$$

in probability as $N \to \infty$ for each bounded and continuous $f$. This convergence also holds simultaneously for a countable set of functions $f$ such as the polynomials in $\tanh(x)$ with rational coefficients. By Skorohod coupling, we may assume that this simultaneous convergence holds a.s., so that we can deduce that the weak convergence $\mu_{\boldsymbol{A}^{(k)}} \to \nu$ holds a.s. as $N \to \infty$. Using that $\boldsymbol{X}$ and $\boldsymbol{A}^{(k)}$ are independent, we now condition on $\boldsymbol{A}^{(k)}$ and apply Lemma 4.3 to infer that the empirical spectral distribution of $\boldsymbol{M}$ converges a. s. in the weak topology as $N \to \infty$ to the free additive convolution $\mu_\beta \boxplus \hat{\nu}$. Without the Skorohod coupling, the empirical spectral distribution of $\boldsymbol{M}$ still converges weakly in distribution to $\mu_\beta \boxplus \hat{\nu}$. The assertion now follows from Lemma 6.1 below. $\square$

The support $\operatorname{supp}\mu$ of a probability measure $\mu$ on $\mathbb{R}$ is defined by

$$\operatorname{supp}\mu := \mathbb{R} \setminus \{t \in \mathbb{R} : \exists \epsilon > 0 \text{ with } \mu(t-\epsilon, t+\epsilon) = 0\}. \tag{6.4}$$

**Lemma 6.1.** *The free additive convolution $\mu_\beta \boxplus \nu$ has support of the form $(-\infty, d]$ with $d < \beta^2(1-q)$ below and above the AT line (i.e. if (1.7) holds with strict inequality or if (1.7) does not hold), and $d = \beta^2(1-q)$ on the AT line (i.e. if (1.7) holds with equality).*

*Proof.* Let $H_{\beta,\nu}(z)$ be defined by (4.6) and $\mathcal{O}_{\beta,\nu}$ by (4.7). From the work of Biane [6], see Proposition 2.2 of [11], we have the equivalence

$$x \in \mathbb{R} \setminus \operatorname{supp}\mu_\beta \boxplus \nu \quad \Leftrightarrow \quad \exists u \in \mathcal{O}_{\beta,\nu} \text{ such that } x = H_{\beta,\nu}(u), \tag{6.5}$$

noting that the proof of Proposition 2.2 of [11] passes through even though our $\nu$ is not compactly supported. Let

$$d := \inf_{u \in \mathcal{O}_{\beta,\nu}} H_{\beta,\nu}(u). \tag{6.6}$$

We note that $\operatorname{supp}\nu = (-\infty, -1]$ and

$$H'_{\beta,\nu}(u) = 1 - \beta^2 \mathsf{E}\left(\frac{1}{(u + \frac{1}{1-\tanh^2(h+\beta\sqrt{q}Z)})^2}\right), \tag{6.7}$$

For $u = 0$, we evaluate

$$H_{\beta,\nu}(0) = \beta^2 \mathsf{E}\left(1 - \tanh^2(h + \beta\sqrt{q}Z)\right) = \beta^2(1-q). \tag{6.8}$$

From (6.7) and as $1 - \tanh^2(x) = \cosh^{-2}(x)$, we can rewrite

$$H'_{\beta,\nu}(0) = 1 - \beta^2 \mathsf{E}\cosh^{-4}(h + \beta\sqrt{q}Z). \tag{6.9}$$

Hence, the AT condition (1.7) is equivalent to $H'_{\beta,\nu}(0) \geq 0$, and that (1.7) with strict inequality is equivalent to $H'_{\beta,\nu}(0) > 0$. Moreover, (6.7) shows that $H'_{\beta,\nu}(u)$ is strictly increasing in $u \in (-1, \infty)$. From (4.6) and as $\operatorname{supp}\nu = (-\infty, -1]$, we obtain that $H_{\beta,\nu}$ exists and is analytic in $(-1, \infty)$.

We first consider $(\beta, h)$ that satisfy (1.7) with strict inequality. Then from $H'_{\beta,\nu}(0) > 0$, we infer that $H_{\beta,\nu}$ attains its infimum over $\mathcal{O}_{\beta,\nu}$ at some $u_* < 0$, and $d = H_{\beta,\nu}(u_*) < H_{\beta,\nu}(0) = \beta^2(1-q)$.

Next, we consider the case hat $(\beta, h)$ does not satisfy (1.7). Then from $H'_{\beta,\nu}(0) < 0$, we infer that $H_{\beta,\nu}$ attains its infimum over $\mathcal{O}_{\beta,\nu}$ at some $u_* > 0$, that $H_{\beta,\nu}$ is decreasing in $(0, u_*)$, and hence $d = H_{\beta,\nu}(u_*) < H_{\beta,\nu}(0) = \beta^2(1-q)$.

For $(\beta, h)$ satisfying (1.7) with equality, we have $H'_{\beta,\nu}(0) = 0$, and $H_{\beta,\nu}$ attains its infimum over $\mathcal{O}_{\beta,\nu}$ at 0, which implies $d = H_{\beta,\nu}(0) = \beta^2(1-q)$. $\square$

## 7. Proof for largest eigenvalue (Theorem 1.5)

*Proof of Theorem 1.5.* As in (5.11), we evaluate the Hessian $\boldsymbol{H}(\boldsymbol{m})$ from (1.13) in $\boldsymbol{m} \in [-1,1]^N$ as

$$\boldsymbol{H}(\boldsymbol{m}) = \frac{\beta}{\sqrt{N}}\bar{\boldsymbol{g}} + \boldsymbol{A}_N - \beta^2(1-q)\mathbf{1} + \eta\mathbf{1} \tag{7.1}$$

where now

$$A_{N,ij} = \frac{-\delta_{ij}}{1-m_i^2} + \frac{2\beta^2}{N} m_i m_j\,, \quad i,j = 1,\dots,N, \tag{7.2}$$

$$\eta = \frac{\beta^2}{N}\sum_{i=1}^N m_i^2 - \beta^2 q\,. \tag{7.3}$$

The assumptions and (1.6) imply $\frac{1}{N}\sum_{i=1}^N m_i^2 \to \mathsf{E}\tanh^2(h+\beta\sqrt{q}Z) = q$ as $N\to\infty$.

First we study the eigenvalues of the matrix $\boldsymbol{A}_N$ via its resolvent. For $u>0$, we define the diagonal matrix $\boldsymbol{D} = \operatorname{diag}\left(u + \frac{1}{1-m_i^2}\right)$, so that the resolvent reads

$$(u\mathbf{1} - \boldsymbol{A}_N)^{-1} = (\boldsymbol{D} - 2\beta^2 \boldsymbol{m}\otimes\boldsymbol{m})^{-1} = \boldsymbol{D}^{-1/2}\left(\mathbf{1} - 2\beta^2(\boldsymbol{D}^{-1/2}\boldsymbol{m})\otimes(\boldsymbol{m}\boldsymbol{D}^{-1/2})\right)^{-1}\boldsymbol{D}^{-1/2}. \tag{7.4}$$

The Sherman-Morrison Lemma [28] thus gives that $u\mathbf{1} - \boldsymbol{A}_N$ is invertible if and only if $2\beta^2 \mathrm{tr}\,\boldsymbol{D}^{-1}\boldsymbol{m}\otimes\boldsymbol{m} \neq 1$. The latter condition is equivalent to

$$\frac{2\beta^2}{N}\sum_{i=1}^N \frac{m_i^2}{u + \frac{1}{1-m_i^2}} \neq 1 \tag{7.5}$$

and also to $\boldsymbol{A}_N$ having an eigenvalue at $u$.

To show (ii), we now assume that $\boldsymbol{m}_N \in \bar{P}_N^{2,\epsilon}$ and let $\epsilon' > 0$. The expression on the left-hand side of (7.5) converges to

$$2\beta^2 \mathsf{E}\left(\frac{\tanh^2(h+\beta\sqrt{q}Z)}{u + \frac{1}{1-\tanh^2(h+\beta\sqrt{q})}}\right) \tag{7.6}$$

as $N\to\infty$. In particular, for $u=0$ the expression in (7.6) is larger than $1+\epsilon$ by definition of $\bar{P}_N^{2,\epsilon}$ and the assumptions. Moreover, (7.6) decreases continuously to 0 as $u\to\infty$, hence it is equal to 1 at some $u>0$. It follows that there exist $0 < u_{N,-} < u_{N,+}$ and $N_0 \in \mathbb{N}$, depending only on $\beta$ and $h$, such that for all $N \geq N_0$, the left-hand side of (7.5) is larger than $1+\epsilon'$ when evaluated at $u = u_{N,-}$, and smaller than than $1-\epsilon'$ when evaluated at $u = u_{N,+}$. As also the expression on the left-hand side of (7.5) is continuous, it follows that it is equal to 1 for some $u_N$ in $(u_{N,-}, u_{N,+})$. As $\epsilon' > 0$ was arbitrary, it follows that $u_N$ converges to the $u>0$ at which the expression in (7.6) is equal to 1. From (7.5), it then follows that $\boldsymbol{A}_N$ has an eigenvalue at $u_N$. As the expression on the left-hand side of (7.5) is decreasing in $u$, it furthermore follows that $\boldsymbol{A}_N$ does not have eigenvalues that are larger than $u_N$. Hence, we have $\lambda_1(\boldsymbol{A}_N) = u_N$.

Let now $u_\infty = \lim_{N\to\infty} u_N$ and $\nu := \mathcal{L}(\frac{-1}{1-\tanh^2(h+\beta\sqrt{q}Z)})$. From the proof of Lemma 6.1, we recall that $H'_{\beta,\nu}(u) > 0$ for all $u>0$. Noting that $u_\infty \in \mathcal{O}_{\beta,\nu}$, and that assumption (4.8) is satisfied as $\operatorname{supp}\nu = (\infty, -1]$, we can now apply Lemma 4.4(i) to obtain

$$\lim_{N\to\infty}\lambda_1\left(\frac{\beta}{\sqrt{N}}\bar{\boldsymbol{g}} + \boldsymbol{A}_N\right) = H_{\beta,\nu}(u_\infty) \quad \text{a.s.} \tag{7.7}$$

As $H_{\beta,\nu}(u_\infty) - \beta^2(1-q) > 0$, assertion (ii) follows.

To show assertion (i), we first note that analogously to the above, there exists $u_\infty > 0$ such that the expression in (7.6) equals 1 at $u = -u_\infty$, and that this implies $\lambda_1(\boldsymbol{A}_N) \to -u_\infty$ as $N\to\infty$. If $-u_\infty \in \mathcal{O}_{\beta,\nu}$, the assertion follows as in (7.7) and as $H_{\beta,\nu}(-u_\infty) <$

$H_{\beta,\nu}(0) = \beta^2(1-q)$. If $-u_\infty \notin \mathcal{O}_{\beta,\nu}$, the assertion follows from Lemma 4.4(ii) and Theorem 1.4. $\square$

## 8. Proof for largest eigenvalue (Theorem 1.2)

In Theorem 1.2, we rely on a specific magnetization $\boldsymbol{m}_N$ at which we evaluate the Hessian of the TAP functional: for $N \in \mathbb{N}$, let $\boldsymbol{v}$ be an eigenvector to the largest eigenvalue of $\beta N^{-1/2}\bar{\boldsymbol{g}}$ with $\|\boldsymbol{v}\|_2 = 1$, then we recall that $\beta N^{-1/2}\boldsymbol{v}^T\bar{\boldsymbol{g}}\boldsymbol{v} \to 2\beta$ a. s. For $\alpha \in [0,1]$ to be chosen later, we define the magnetization $\boldsymbol{m}_N^\alpha$ by

$$m_{N,i}^\alpha = \alpha \operatorname{sign}(v_i), \quad i = 1, \ldots, N. \tag{8.1}$$

First we note that for $\beta > 0$ and $\alpha^2 > 1 - 1/\beta$,

$$\frac{\beta^2}{N}\sum_{i=1}^N \left(1 - {m_{N,i}^\alpha}^2\right)^2 = \beta^2\left(1-\alpha^2\right)^2 < 1, \tag{8.2}$$

and thus $\boldsymbol{m}_N^\alpha \in P_N^1$.

*Proof of Theorem 1.2.* Let $\boldsymbol{m}_N = \boldsymbol{m}_N^\alpha$ and $\boldsymbol{v}$ be defined by (8.1). As in (5.11), we evaluate $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{m}_N)$ as follows:

$$\begin{aligned} H_{ij} &= \frac{\beta}{\sqrt{N}}\bar{g}_{ij} + \frac{2\beta^2\alpha^2}{N}\operatorname{sign}(v_i)\operatorname{sign}(v_j), \quad i,j = 1,\ldots,N, i \neq j \\ H_{ii} &= -\beta^2\left(1-\alpha^2\right) - \frac{1}{1-\alpha^2} + \frac{2\beta^2\alpha^2}{N}. \end{aligned} \tag{8.3}$$

We now estimate $\boldsymbol{v}^T\boldsymbol{H}\boldsymbol{v}$ which is a lower bound for $\lambda_1(\boldsymbol{H})$. First, recall that $\boldsymbol{v}^T\frac{\beta}{\sqrt{N}}\bar{\boldsymbol{g}}\boldsymbol{v} \to 2\beta$ a. s. as $N \to \infty$. The random vector $\boldsymbol{v}$ is distributed as the first column of a Haar distributed random matrix on the orthogonal group on $\mathbb{R}^N$ (see e. g. Corollary 2.5.4 in [3]). Hence, by Lemma 8.1 below,

$$\frac{2\beta^2\alpha^2}{N}\sum_{i,j=1}^N v_i \operatorname{sign}(v_i)\operatorname{sign}(v_j)\, v_j \to \frac{4\beta^2\alpha^2}{\pi} \tag{8.4}$$

in probability as $N \to \infty$. It follows that

$$\boldsymbol{v}^T\boldsymbol{H}\boldsymbol{v} \to 2\beta - \beta^2\left(1-\alpha^2\right) - \frac{1}{1-\alpha^2} + \frac{4\beta^2\alpha^2}{\pi} \tag{8.5}$$

in probability as $N \to \infty$. For fixed $\beta > 0$, the expression on the r.h.s. attains its maximum at $\alpha^2 = 1 - \beta^{-1}(1+4/\pi)^{-1/2}$ which is larger than $1-\beta^{-1}$ and hence $\boldsymbol{m}^\alpha \in P_N^1$ by (8.2). The value of the maximum of the r.h.s. of (8.5) is strictly positive for $\beta > \frac{\pi}{2}\left(\sqrt{1+4/\pi} - 1\right) =: \beta_0 \approx 0.798$. $\square$

**Lemma 8.1.** *Let $\boldsymbol{v}$ be distributed as the first column of a Haar distributed random matrix on the orthogonal group on $\mathbb{R}^N$. Then,*

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N |v_i| \to \sqrt{2/\pi} \tag{8.6}$$

*in probability as $N \to \infty$.*

*Proof.* First we consider the expectation

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbb{E}\left|v_i\right| = \mathbb{E}\sqrt{N}\left|v_1\right|, \tag{8.7}$$

which converges to $\mathbb{E}|Z| = \sqrt{2/\pi}$ by (1) of Diaconis and Freedman [15], noting that convergence in total variation implies convergence of (absolute) moments. Likewise, for the second moment, we have

$$\frac{1}{N}\sum_{i,j=1}^{N}\mathbb{E}\left|v_i\right|\left|v_j\right| = (N-1)\mathbb{E}\left|v_1\right|\left|v_2\right| + \mathbb{E}v_1^2. \tag{8.8}$$

Here the second term on the r.h.s. converges to zero, and the first term to $(\mathbb{E}|Z|)^2$ again by (1) of [15]. This shows that the variance of the expression on the l.h.s. of (8.6) converges to zero, so that the convergence of the expectation implies the assertion. □

## Appendix A. Plefka's expansion

We discuss here the relation between the TAP free energy and the free energy: for finite $N$, the TAP free energy can be interpreted in terms of an expansion of the Gibbs potential of the SK model [27]. In the following, we give an introduction of the approach based on [21]: For $\alpha \in \mathbb{R}$, and $\boldsymbol{\varphi} = \{\varphi_i\}_{i=1,\dots,N} \in \mathbb{R}^N$, we define the Hamiltonian

$$H_{\alpha,\beta,h,\boldsymbol{\varphi}}(\sigma) = \alpha H_{\beta,0}(\sigma) + h\sum_{i\le N}\sigma_i + \sum_{i\le N}\varphi_i\sigma_i, \tag{A.1}$$

the partition function

$$Z_{\alpha,\beta,h,\boldsymbol{\varphi}} = 2^{-N}\sum_{\sigma\in\Sigma_N}\exp H_{\alpha,\beta,h,\boldsymbol{\varphi}} \tag{A.2}$$

and the (normalized) functional $G_N(\alpha,\boldsymbol{\varphi})$ by

$$G_N(\alpha,\boldsymbol{\varphi}) := \log Z_{\alpha,\beta,h,\boldsymbol{\varphi}}. \tag{A.3}$$

Note that the map $\boldsymbol{\varphi} \mapsto G_N(\alpha,\boldsymbol{\varphi})$ is *convex* as it is a concatenation of convex functions. In particular, the Legendre transform is well defined:

$$G_N^\star(\alpha,\boldsymbol{m}) = \sup_{\boldsymbol{\varphi}\in\mathbb{R}^N}\left\{\sum_{i\le N}\varphi_i m_i - G_N(\alpha,\boldsymbol{\varphi})\right\}. \tag{A.4}$$

Again by convexity, the $^\star$-operation is an *involution*, i.e. with the property that $G_N = (G_N^\star)^\star$. Since by construction $N^{-1}G_N(1,\boldsymbol{0})$ coincides with the free energy, we therefore have that

$$F_N(\beta,h) = \frac{1}{N}\sup_{\boldsymbol{m}\in[-1,1]^N}\left\{-G_N^\star(1,\boldsymbol{m})\right\}. \tag{A.5}$$

Here the supremum can be taken over $\boldsymbol{m} \in [-1,1]^N$ as it is readily checked that $G_N^\star(1,\boldsymbol{m}) < \infty$ only for these $\boldsymbol{m}$. The thermodynamic variables $\boldsymbol{m} \in \mathbb{R}^N$ are dual to the magnetic fields $\boldsymbol{\varphi}$, and correspond, after closer inspection, to the *magnetization*: indeed, given a function $f$, we denote by

$$\langle f\rangle_\alpha := \frac{2^{-N}\sum_\sigma f(\sigma)\exp\left(H_{\alpha,\beta,h,\boldsymbol{\varphi}}(\sigma)\right)}{Z_{\alpha,\beta,h,\boldsymbol{\varphi}}}, \tag{A.6}$$

the Gibbs expectation with respect to the Hamiltonian appearing in (A.1), and one immediately checks by solving the variational principle (A.4) that the fundamental relation

$$\langle\sigma_i\rangle_\alpha = m_i \tag{A.7}$$

holds. In particular, we see from the above that $m_i \in [-1,1]$. The idea is to now proceed by Taylor expansion of the Gibbs potential,

$$-G^\star_N(\alpha,\boldsymbol{m}) = \sum_{k=0}^\infty \frac{\partial^k}{\partial\alpha^k}\Big(-G^\star_N(\alpha,\boldsymbol{m})\Big)\Big|_{\alpha=0}\frac{\alpha^k}{k!}, \tag{A.8}$$

and to evaluate this in $\alpha=1$, provided the expansion converges. The calculation of the Taylor coefficients considerably simplifies in $\alpha=0$, as one only needs to compute "spin-correlations" under the non-interacting Hamiltonian $H_{0,\beta,h,\boldsymbol{\varphi}}(\sigma)=\sum_{i\le N}(\varphi_i+h)\sigma_i$. First note that for $\alpha=0$, the variational principle (A.4) is solved by $\boldsymbol{\varphi}^*$ such that

$$m_i = \langle\sigma_i\rangle_0 = \tanh\left(h+\varphi_i^*\right). \tag{A.9}$$

One immediately checks that the $0^{th}$-term of the expansion is given by

$$\begin{aligned}
-G^\star_N(0,\boldsymbol{m}) &= -\sum_{i\le N}\left(\varphi_i^* m_i - \log\cosh\left(h+\varphi_i^*\right)\right)\\
&= -\sum_{i\le N}\tanh^{-1}(m_i)m_i + h\sum_{i\le N}m_i + \sum_{i\le N}\log\cosh\left(\tanh^{-1}(m_i)\right)\\
&= h\sum_{i\le N}m_i - \sum_{i\le N}I(m_i),
\end{aligned} \tag{A.10}$$

where we used (A.9) for the second line and (1.9) for the third line.

For the first derivative in $\alpha=0$, we have

$$\begin{aligned}
-\frac{\partial}{\partial\alpha}G^\star_N(\alpha,\boldsymbol{m})\Big|_{\alpha=0} &= \left\langle \frac{\beta}{\sqrt{2N}}\sum_{i,j\le N} g_{ij}\sigma_i\sigma_j\right\rangle_0\\
&= \frac{\beta}{\sqrt{2N}}\sum_{i\ne j\le N} g_{ij}\langle\sigma_i\rangle_0\langle\sigma_j\rangle_0 + \frac{\sqrt{2}\beta}{\sqrt{N}}\sum_{i\le N} g_{ii}\left\langle\sigma_i^2\right\rangle_0\\
&= \frac{\beta}{\sqrt{N}}\sum_{i<j\le N}\bar g_{ij}m_im_j + N\times o_N(1),
\end{aligned} \tag{A.11}$$

where we used the fact that $\sigma_i$ and $\sigma_j$ are independent under the Gibbs measure for $\alpha=0$, and $o_N(1)\to0$ as $N\to\infty$. The second order term in (A.8) is left to the reader but one can check that

$$-\frac{\partial^2}{\partial\alpha^2}G^\star_N(\alpha,m)\Big|_{\alpha=0} = \frac{\beta^2}{N}\sum_{i<j}^N \bar g_{ij}^2(1-m_i^2)(1-m_j^2) + N\times o_N(1). \tag{A.12}$$

This computation is done in [27] by Plefka. All in all, we obtain

$$F_N(\beta,h)=\sup_{\boldsymbol{m}\in[-1,1]^N}\left\{\frac{1}{N}\mathrm{TAP}^*_N(\boldsymbol{m})+o_N(1)+\frac{1}{N}\sum_{k=3}^{\infty}\frac{\partial^k}{\partial\alpha^k}\Big(-G_N(\alpha,\boldsymbol{m})^\star\Big)\Big|_{\alpha=0}\frac{1}{k!}\right\}, \tag{A.13}$$

with

$$\mathrm{TAP}^*_N(\boldsymbol{m})=\frac{\beta}{\sqrt{N}}\sum_{i<j\le N}\bar{g}_{ij}m_im_j+h\sum_{i=1}^N m_i+\frac{\beta^2}{2N}\sum_{i<j}^N\bar{g}_{ij}^2(1-m_i^2)(1-m_j^2)-\sum_{i=1}^N I(m_i). \tag{A.14}$$

Note that the argument in the supremum in (A.13) coincides with the Legendre transform $-G^\star_N(1,\boldsymbol{m})$ by (A.5) and thus has to be concave. By replacing $\bar{g}_{ij}^2$ by one in (A.14), like Plefka does in [27], we obtain $\mathrm{TAP}_N(\boldsymbol{m})$. The problem of this approach is to justify when the Taylor expansion (A.8) exists and the $\frac{1}{N}\sum_{k=3}^{\infty}\frac{d^k}{d\alpha^k}\Big(-G^\star_N(\alpha,m)\Big)\Big|_{\alpha=0}\frac{1}{k!}$ term is negligible.

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