Gradient flow and functional inequalities for quantum Markov semigroups, II

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Detailed balance

Let $P_{i,j}$ be the Markov transition matrix for a Markov chain on a finite state space $S = \{x_1, \ldots, x_n\}$. Suppose that σ is a probability density on S with

$$\sigma_j = \sum_{i=1}^{n} \sigma_i P_{i,j} \ .$$

The transition matrix satisfies the *detailed balance condition with respect to* σ in case

$$\sigma_i P_{i,j} = \sigma_j P_{j,i} \qquad \text{for all } i, j \ .$$

The matrix P is self-adjoint on $\mathbb{C}^n$ equipped with the inner product

$$\langle f, g\rangle_\sigma = \sum_{k=1}^{n} \sigma_k \overline{f_k} g_k \ ,$$

if and only if the detailed balance condition is satisfied.

There are a number of different ways one might generalize this inner product to the quantum setting, and these give different notions of self-adjointness.

Definition 0.1 (Compatible inner product). *An inner product* $\langle \cdot, \cdot \rangle$ *is* compatible *with* $\sigma \in \mathfrak{S}_+(\mathcal{A})$ *in case for all* $A \in \mathcal{A}$*,* $\mathrm{Tr}[\sigma A] = \langle \mathbf{1}, A \rangle$*.*

If a quantum Markov semigroup $\mathscr{P}_t$ is self-adjoint with respect to an inner product $\langle \cdot, \cdot \rangle$ that is compatible with $\sigma \in \mathfrak{S}_+$, then for all $A \in \mathcal{A}$,

$$\mathrm{Tr}[\sigma A] = \langle \mathbf{1}, A \rangle = \langle \mathscr{P}_t \mathbf{1}, A \rangle = \langle \mathbf{1}, \mathscr{P}_t A \rangle = \mathrm{Tr}[\sigma \mathscr{P}_t A] \ ,$$

and thus σ is invariant under $\mathscr{P}_t^\dagger$.

Definition 0.2. *Let* $\sigma \in \mathfrak{S}_+$ *be a non-degenerate density matrix. For each* $s \in \mathbb{R}$*, and each* $A, B \in \mathcal{A}$*, define*

$$\langle A, B \rangle_s = \mathrm{Tr}[(\sigma^{(1-s)/2} A \sigma^{s/2})^* (\sigma^{(1-s)/2} B \sigma^{s/2})] = \mathrm{Tr}[\sigma^s A^* \sigma^{1-s} B] \, .$$

Definition 0.3 (Modular operator and modular group). *Let* $\sigma \in \mathfrak{S}_+$. *Define a linear operator* Δ_σ *on* $\mathfrak{H}_{\mathcal{A}}$*, or, what is the same thing, on* $\mathcal{A}$*, by*

$$\Delta_\sigma(A) = \sigma A \sigma^{-1} .$$

Δ_σ *is called the* modular operator. *The* modular generator *is the self-adjoint element* $h \in \mathcal{A}$ *given by*

$$h = -\log \sigma ,$$

The modular automorphism group α_t *on* $\mathcal{M}_n(\mathbb{C})$ *is the group defined by*

$$\alpha_t(A) = e^{ith} A e^{-ith} .$$

Note that $\Delta_\sigma = \alpha_i$.

Let $\sigma \in \mathfrak{S}_+$ and note that

$$\mathrm{Tr}[A^*\Delta_\sigma B] = \mathrm{Tr}[(\Delta_\sigma A)^* B] \quad \text{and} \quad \mathrm{Tr}[A^*\Delta_\sigma A] = \mathrm{Tr}[|\sigma^{1/2}A\sigma^{-1/2}|^2]$$

so that Δ_σ is a positive operator on $\mathfrak{H}_\mathcal{A}$.

Since Δ_σ is strictly positive, all eigenvalues of Δ_σ are strictly positive, hence we may write them in the form $e^{-\omega_\gamma}$. Since $(\Delta_\sigma A)^* = \Delta_\sigma^{-1}A^*$, it follows that for all $E \in \mathfrak{H}_\mathcal{A}$,

$$\Delta_\sigma E = e^{-\omega}E \quad \Longleftrightarrow \quad \Delta_\sigma E^* = e^{\omega}E^* \ .$$

The following is due to Alicki:

Lemma 0.4. *Let $\sigma \in \mathfrak{S}_+$ be a non-degenerate density matrix, and let $s \in [0,1]$, $s \neq 1/2$. Let $\mathscr{K}$ be any operator on $\mathcal{A}$ that is self-adjoint with respect to $\langle \cdot, \cdot \rangle_s$ and also preserves self-adjointness. Then $\mathscr{K}$ commutes with α_t, for all t, real and complex.*

Definition 0.5 (Detailed balance)**.** *A QMS $\mathscr{P}_t$ on $\mathcal{A}$ satisfies the* detailed balance condition *with respect to $\sigma \in \mathfrak{S}_+(\mathcal{A})$ in case for each $t > 0$, $\mathscr{P}_t$ is self-adjoint in the σ-GNS inner product $\langle \cdot, \cdot \rangle_1$. In this case σ is invariant under $\mathscr{P}_t^\dagger$, and we say that the QMS $\mathscr{P}_t$ satisfies the σ-DBC.*

$$\langle A, B \rangle_s = \mathrm{Tr}[\sigma^s A^* \sigma^{1-s} B] = \langle A, \Delta_\sigma^s B \rangle_1 \ .$$

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Hence if $\mathscr{P}_t$ satisfies the σ-DBC

$$\begin{aligned} \langle A, \mathscr{P}_t B\rangle_s &= \langle A, \Delta_\sigma^s \mathscr{P}_t B\rangle_1 = \langle A, \mathscr{P}_t \Delta_\sigma^s B\rangle_1 \\ &= \langle \mathscr{P}_t A, \Delta_\sigma^s B\rangle_1 = \langle \mathscr{P}_t A, B\rangle_s \ . \end{aligned}$$

In particular, if $\mathscr{P}_t$ satisfies the σ-DBC, for each t, $\mathscr{P}_t$ self adjoint with respect to any of the inner products $\langle \cdot, \cdot \rangle_s$, or, more genraly any of the inner products

$$\langle A, B\rangle_\mu = \int_0^1 \langle A, B\rangle_s \mathrm{d}\mu \ .$$

$\langle \cdot, \cdot \rangle_{1/2}$ is special; this is the KMS inner product.

A QMS $\mathscr{P}_t = e^{t\mathscr{L}}$ on $\mathcal{A}$ that satisfies the σ-DBC for $\sigma \in \mathfrak{S}_+(\mathcal{A})$ has a generator $\mathscr{L}$ that commutes with the modular operator Δ_σ. Hence Δ_σ and $\mathscr{L}$ can be simultaneously diagonalized.

In the case $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$, the diagonalization of Δ_σ reduces immediately to the diagonalization of σ: Let $\sigma = e^{-h}$ be a density matrix on $\mathbb{C}^n$. Let $\{\eta_1, \dots, \eta_n\}$ be an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of $h = -\log \sigma$:

$$h\eta_j = \lambda_j \eta_j \ .$$

For $\alpha = (\alpha_1, \alpha_2) \in \{(i,j) \ : 1 \leq i,j \leq n\}$, define numbers ω_α (called the *Bohr frequencies*) by

$$\omega_\alpha = \lambda_{\alpha_1} - \lambda_{\alpha_2} \ ,$$

and rank-one operators F_α given by $F_\alpha = |\eta_{\alpha_1}\rangle\langle\eta_{\alpha_2}|$ where for $\eta, \xi \in \mathbb{C}^n$, $|\eta\rangle\langle\xi|$ is the rank-one operator sending ζ to $\langle\xi, \zeta\rangle_{\mathbb{C}^n}\eta$. Evidently

$$\Delta_\sigma F_\alpha = e^{-\omega_\alpha} F_\alpha \qquad \text{and} \qquad F_\alpha^* = F_{\alpha'} \quad \text{where} \quad \alpha' = (\alpha_2, \alpha_1) \ .$$

Theorem 0.6. *Let* $\mathscr{P}_t = e^{t\mathscr{L}}$ *be a QMS on* $\mathcal{M}_n(\mathbb{C})$ *that satisfies the* σ*-DBC for* $\sigma \in \mathfrak{S}_+$*. Then the generator* $\mathscr{L}$ *of* $\mathscr{P}_t$ *has the form*

$$\mathscr{L}A = \sum_{j\in\mathcal{J}} \left(e^{-\omega_j/2} V_j^*[A, V_j] + e^{\omega_j/2}[V_j, A]V_j^* \right) ,$$

where:

(i) $\tau[V_j^* V_k] = c_j \delta_{j,k}$ *for all* $j, k \in \mathcal{J}$*. (ii)* $\tau[V_j] = 0$ *for all* $j \in \mathcal{J}$*. (iii)* $\{V_j\}_{j\in\mathcal{J}} = \{V_j^*\}_{j\in\mathcal{J}}$*. (iv)* $\{V_j\}_{j\in\mathcal{J}}$ *consists of eigenvectors of the modular operator* Δ_σ *with*

$$\Delta_\sigma V_j = e^{-\omega_j} V_j \ .$$

Conversely, given any set any set $\{V_j\}_{j\in\mathcal{J}}$ satisfying *(i)*, *(ii)*, *(iii)*, the operator $\mathscr{L}$ given by this formula is the generator of a QMS $\mathscr{P}_t$ that satisfies the σ-DBC.

The fact that the operators V_j, $j \in \mathcal{J}$ are eigenfunctions of Δ_σ, and hence Δ_σ^s for all s, has the following consequence:

$$\sigma^s V_j = \sigma^s V_j \sigma^{-s}\sigma^s = (\Delta^s V_j)\sigma^s = e^{-s\omega_j} V_j \sigma^s \ .$$

Differentiating in s at $s = 0$,

$$[V_j, h] = -\omega_j V_j \ .$$

Non-commutative derivatives

Fix such a generator $\mathscr{L}$, and the sets $\{V_j\}_{j\in\mathcal{J}}$ and $\{\omega_j\}_{j\in\mathcal{J}}$ as above.

Define operators ∂_j on $\mathcal{A}$ by

$$\partial_j A = [V_j, A] \quad \text{so that} \quad \partial_j^\dagger A = [V_j^*, A]\ .$$

Define an operator $\mathscr{L}_0$ on $\mathfrak{H}_\mathcal{A}$ by

$$\mathscr{L}_0 A = -\sum_{j\in\mathcal{J}} \partial_j^\dagger \partial_j A = -\sum_{j\in\mathcal{J}} [V_j^*, [V_j, A]]\ .$$

We may write $\mathscr{L}_0 A = -\sum_{j\in\mathcal{J}} (V_j^*[V_j, A] + [A, V_j]V_j^*)$, and

hence $\mathscr{L}_0$ is the generator of QMS.

Define the Hilbert space $\mathfrak{H}_{\mathcal{A},\mathcal{J}}$ by

$$\mathfrak{H}_{\mathcal{A},\mathcal{J}} = \bigoplus_{j\in\mathcal{J}} \mathfrak{H}_{\mathcal{A}}^{(j)} \ ,$$

where each $\mathfrak{H}_{\mathcal{A}}^{(j)}$ is a copy of $\mathfrak{H}_{\mathcal{A}}$. For $\mathbf{A} \in \mathfrak{H}_{\mathcal{A},\mathcal{J}}$ and $j \in \mathcal{J}$, let A_j denote the component of $\mathbf{A}$ in $\mathfrak{H}_{\mathcal{A}}^{(j)}$. Thus, picking some linear ordering of $\mathcal{J}$, we can write

$$\mathbf{A} = (A_1, \ldots, A_{|\mathcal{J}|}) \ .$$

Define an operator $\nabla : \mathfrak{H}_{\mathcal{A}} \to \mathfrak{H}_{\mathcal{A},\mathcal{J}}$ by

$$\nabla A = (\partial_1 A, \ldots, \partial_{|\mathcal{J}|} A) \ .$$

We define the operator $\mathrm{div} : \mathfrak{H}_{\mathcal{A},\mathcal{J}} \to \mathfrak{H}_{\mathcal{A}}$ by

$$\mathrm{div}\,\mathbf{A} = -\sum_{j\in\mathcal{J}} \partial_j^\dagger A_j = \sum_{j\in\mathcal{J}} [A_j, V_j^*] \ .$$

Note that div is minus the adjoint of the map $\nabla : \mathfrak{H}_{\mathcal{A}} \to \mathfrak{H}_{\mathcal{A},\mathcal{J}}$, so that $\mathscr{L}_0$ is negative semi-definite. With these definitions, $\mathscr{L}_0 = \mathrm{div} \circ \nabla$. We call ∇ the *non-commutative gradient* associated to $\mathscr{L}$, and div the *non-commutative divergence* associated to $\mathscr{L}$.

Note that each ∂_j is a derivation: For all A, B,

$$\partial_j(AB) = (\partial_j A)B + A\partial_j(B) \ .$$

Lemma 0.7. *For all* $s \in [0,1]$, *all* $j \in \mathcal{J}$, *and all* $A, B \in \mathcal{A}$ *we have*

$$\langle \partial_j B, A \rangle_s = \langle B, e^{s\omega_j}(e^{-\omega_j} V_j^* A - A V_j^*) \rangle_s \ .$$

Consequently, for all $s \in [0,1]$, and all $A, B \in \mathcal{A}$,

$$e^{(1/2-s)\omega_j} \langle \partial_j B, \partial_j A \rangle_s = -\langle B, e^{-\omega_j/2} V_j^* [A, V_j] + e^{\omega_j/2} [V_j, A] V_j^* \rangle_s \ .$$

Therefore $\mathcal{E}_s(B, A) = -\langle B, \mathscr{L} A \rangle_s$

$$\mathcal{E}_s(B, A) := \sum_{j \in \mathcal{J}} c_j e^{(1/2-s)\omega_j} \langle \partial_j B, \partial_j A \rangle_s \ .$$

Chain rule

The evolution equation

$$\frac{\partial}{\partial t}\rho(x,t) = \nabla \cdot (\rho(x,t)[\nabla \log \rho(x,t) - \nabla \log \sigma(x)])$$

is a linear equation because of the chain rule identity

$$\rho \nabla \log \rho = \nabla \rho \ .$$

To obtain a non-commutartive analog, write

$$\rho = \lim_{n\to\infty} \left(\mathbf{1} + \frac{1}{n} \log \rho \right)^n \ .$$

for any V,

$$[V, \rho] = \lim_{n\to\infty} \sum_{m=0}^{n-1} \frac{1}{n} \left(\mathbf{1} + \frac{1}{n}\log\rho\right)^m [V, \log\rho] \left(\mathbf{1} + \frac{1}{n}\log\rho\right)^{n-m-1}$$
$$= \int_0^1 \rho^s [V, \log\rho] \rho^{1-s} \mathrm{d}s \ .$$

The operation $A \mapsto \int_0^1 \rho^s A \rho^{1-s} \mathrm{d}s = R_\rho \int_0^1 \Delta_\rho^s A \mathrm{d}s$, where R_ρ is right multiplication by ρ, is a non-commutative anolog of multiplication by ρ, and it takes self-adjoint operators to self-adjoint operators.

The map

$$A \mapsto \int_0^1 \rho^s A \rho^{1-s} \mathrm{d}s = R_\rho \int_0^1 \Delta_\rho^s(A) \mathrm{d}s =: R_\rho f_0(\Delta_\rho)(A) \ .$$

has the inverse

$$A \mapsto \int_0^\infty \frac{1}{t+\rho} A \frac{1}{t+\rho} \mathrm{d}t \ .$$

These are two natural quantum analogs of "multiplication by ρ" and "division by ρ" that fequently arise in the study of quantum systems.

There is another useful way to look at this pair of linear transformations. By the Spectral Theorem, if A is a strictly positive operator,

$$\log A = \int_0^\infty \left(\frac{1}{1+t} - \frac{1}{A+t} \right) \mathrm{d}t \ .$$

Hence for A self-adjoint

$$\lim_{h\to 0} \frac{1}{h}(\log(A+hH) - \log(A)) = \int_0^1 \frac{1}{A+t} H \frac{1}{A+t} \mathrm{d}t \ .$$

Hence "quantum division by A" is the derivative of the logarithm function.

By Duhamel's Formula, for K and H self-adjoint,

$$e^{K+hH} = \int_0^1 e^{s(K+hH)} hHe^{(1-s)(K)} \mathrm{d}s + e^K \ .$$

Hence

$$\lim_{h\to 0} \frac{1}{h}(e^{K+hH} - e^K) = \int_0^1 e^{sK} He^{(1-s)K} \mathrm{d}s \ .$$

That is, "quantum multiplication of e^K by H" is

$$\frac{\mathrm{d}}{\mathrm{d}h} e^{K+hH}\bigg|_{h=0} \ .$$

This identification has an important consequence. Let v be a unit vector in $\mathcal{H}$, and let $|v\rangle\langle v|$ denote the orthogonal projection onto the span of v. It is easy to see that if v is not an eigenvector of A, then

$$B := \int_0^1 \frac{1}{A+t}|v\rangle\langle v|\frac{1}{A+t}\mathrm{d}t$$

is not a rank-one operator. Let

$$B = \sum_{j=1}^m \lambda_j |u_j\rangle\langle u_j|$$

be a spectral resolution of B.

Then

$$|v\rangle\langle v| = \sum_{j=1}^{m} \lambda_j \int_0^1 A^{1-s}|u_j\rangle\langle u_j|A^s \mathrm{d}s \ .$$

An invertible linear map from $M_n(\mathbb{C}^n)$ to $M_n(\mathbb{C}^n)$ cannot send two $|u_j\rangle\langle u_j|$ and $|u_j\rangle\langle u_j|$ to multiples of $|v\rangle\langle v|$ for distinct j and k, and this ensures that for some j with $\lambda_j > 0$,

$$\int_0^1 A^{1-s}|u_j\rangle\langle u_j|A^s \mathrm{d}s$$

is not positive. That is, the matrix exponential function is non monotone: It is not necessarily the case that for H self-adjoint and $A > 0$, $e^{H+A} - e^H \geq 0$.

We need some variants on this "multiplication by ρ":
Consider the function f_ω defined by

$$f_\omega(t) := \int_0^1 e^{\omega(s-1/2)} t^s \,\mathrm{d}s = e^{\omega/2} \frac{t - e^{-\omega}}{\log t + \omega} \ .$$

Definition 0.8. *For $\rho \in \mathfrak{S}_+$, and $\omega \in \mathbb{R}$, define the operator $[\rho]_\omega : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ by*

$$[\rho]_\omega = R_\rho \circ f_\omega(\Delta_\rho)$$

For each ω, $[\rho]_\omega$ is invertible, and its inverse, $[\rho]_\omega^{-1} = (1/f_\omega)(\Delta_\rho) \circ R_{\rho^{-1}}$ may then be viewed as the corresponding non-commutative form of division by ρ.

Simple lemmas say:

$$R_\rho f_\omega(\Delta_\rho)\Big(V\log(e^{-\omega/2}\rho) - \log(e^{\omega/2}\rho)V\Big) = e^{-\omega/2}V\rho - e^{\omega/2}\rho V\ .$$

For $\omega = 0$, and $V = V_j$, this is

$$R_\rho f_0(\Delta_\rho)(\partial_j \log\rho) = \partial_j\rho\ .$$

Moreover,

$$\partial_j(\log\rho - \log\sigma) = V_j\log(e^{-\omega_j/2}\rho) - \log(e^{\omega_j/2}\rho)V_j\ .$$

Combining, we can write $\mathscr{L}^\dagger\rho = \sum_{j\in\mathcal{J}}[V^*, e^{-\omega/2}V\rho - e^{\omega/2}\rho V]$

in terms of $D(\rho||\sigma)$.

Theorem 0.9. *Let $\mathscr{P}_t = e^{t\mathscr{L}}$ be QMS on $\mathcal{A}$ that satisfies the σ-DBC for $\sigma \in \mathfrak{S}_+(\mathcal{A})$, and let $\mathscr{L}$ be given in standard from. Then, for all $\rho \in \mathfrak{S}_+$,*

$$-\mathscr{L}^\dagger \rho = \sum_{j \in \mathcal{J}} \partial_j^\dagger \Big([\rho]_{\omega_j} \partial_j (\log \rho - \log \sigma) \Big) \ .$$

We have now arrived at a quantum analog of the classical Kolmogorov forward equation The evolution equation

$$\frac{\partial}{\partial t} \rho(x,t) = \nabla \cdot (\rho(x,t)[\nabla \log \rho(x,t) - \nabla \log \sigma(x)])$$

This is the Kolmogorov forward equation for a diffusion process.

Lemma 0.10 (Chain rule identity)**.** *For all* $V \in \mathcal{M}_n(\mathbb{C})$, $\rho \in \mathfrak{S}_+$ *and* $\omega \in \mathbb{R}$,

$$\int_0^1 e^{\omega(s-1/2)} R_\rho \Delta_\rho^s \Big(V \log(e^{-\omega/2}\rho) - \log(e^{\omega/2}\rho) V \Big)\, \mathrm{d}s = e^{-\omega/2} V\rho - e^{\omega/2} \rho V \ .$$

Proof. Define $f(s) = e^{\omega(1/2-s)} \rho^{1-s} V \rho^s$. The right side equals $f(1) - f(0)$ and

$$\begin{aligned} f'(s) &= e^{\omega(1/2-s)} \rho^{1-s} \Big(-\omega V - \log(\rho) V + V \log(\rho) \Big) \rho^s \\ &= e^{\omega(1/2-s)} \rho^{1-s} \Big(V \log(e^{-\omega/2}\rho) - \log(e^{\omega/2}\rho) V \Big) \rho^s \ . \end{aligned}$$

□

Lemma 0.11. *Let $\mathscr{P}_t = e^{t\mathscr{L}}$ be QMS on $\mathcal{A}$ that satisfies the σ-DBC for $\sigma \in \mathfrak{S}_+(\mathcal{A})$. and let $\mathscr{L}$ be given in standard from. Then for all $\rho \in \mathfrak{S}_+$, and all $j \in \mathcal{J}$,*

$$\partial_j(\log\rho - \log\sigma) = V_j \log(e^{-\omega_j/2}\rho) - \log(e^{\omega_j/2}\rho)V_j\ .$$

Proof. Since $\Delta_\sigma^s V_j = e^{-s\omega_j}V_j$, $[V_j, \log\sigma] = \omega_j V_j$. It follows that

$$\partial_j(\log\rho - \log\sigma) = [V_j, \log\rho] - \omega_j V_j =$$
$$V_j \log(e^{-\omega_j/2}\rho) - \log(e^{\omega_j/2}\rho)V_j\ ,$$

which is the desired identity. □

Riemannian metrics on $\mathfrak{S}$.

We now turn to the construction of a Riemannian metric $g_{\mathscr{L}}$ on $\mathfrak{S}$ for which the quantum equation is gradient flow for the relative entropy.

Let $\rho(t), t \in (t_0, t_1)$, be any differentiable path in $\mathfrak{S}_+$ regarded as a convex subset of $\mathcal{A}$. For each $t \in (t_0, t_1)$, let $\dot{\rho}(t) \in \mathcal{A}$ denote the derivative of $\rho(t)$ in t. If $\rho(t)$ is any differentiable path in $\mathfrak{S}_+$ defined on $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ such that $\rho(0) = \rho_0$, then $\mathrm{Tr}[\dot{\rho}(0)] = 0$, so that there is an affine subspace of $\mathfrak{H}_{\mathcal{A},\mathcal{J}}$ consisting of elements $\mathbf{A}$ for which

$$\dot{\rho}(0) = \operatorname{div} \mathbf{A} \ .$$

We wish to rewrite this as an analog of the classical continuity equation for the time evolutions of a probability density $\rho(x,t)$ on $\mathbb{R}^n$:

$$\frac{\partial}{\partial t}\rho(x,t) + \mathrm{div}[\mathbf{v}(x,t)\rho(x,t)] = 0 \ .$$

In the classical case, for ρ strictly positive, any expression of the form

$$\frac{\partial}{\partial t}\rho(x,t) = \mathrm{div}[\mathbf{a}(x,t)]$$

gives rise to a continuity equation with $\mathbf{v}(x,t) = -\rho^{-1}(x,t)\mathbf{a}(x,t)$. In the quantum case, there are many different ways to multiply and divide by $\rho \in \mathfrak{S}_+$.

Definition 0.12. *Let* $\vec{\omega} \in \mathrm{R}^{|\mathcal{J}|}$*. For* $\rho \in \mathfrak{S}_+$ *we define the linear operator* $[\rho]_{\vec{\omega}}$ *on* $\mathfrak{H}_{\mathcal{A},\mathcal{J}}$ *by*

$$[\rho]_{\vec{\omega}}\big(A_1, \ldots, A_{|\mathcal{J}|}\big) = \big([\rho]_{\omega_1} A_1, \ldots, [\rho]_{\omega_{|\mathcal{J}|}} A_{|\mathcal{J}|}\big) \ .$$

Note that $[\rho]_{\vec{\omega}}$ is invertible with

$$[\rho]_{\vec{\omega}}^{-1}\big(A_1, \ldots, A_{|\mathcal{J}|}\big) = \big([\rho]_{\omega_1}^{-1} A_1, \ldots, [\rho]_{\omega_{|\mathcal{J}|}}^{-1} A_{|\mathcal{J}|}\big) \ .$$

where we have used the fact that R_ρ and Δ_ρ commute. If $\vec{\omega} \in \mathrm{R}^{|\mathcal{J}|}$ is the vector of Bohr frequencies associated to Δ_σ, then for $\mathbf{V} \in \mathfrak{H}_{\mathcal{A},\mathcal{J}}$, define

$$\|\mathbf{V}\|_{\sigma,\rho}^2 = \langle \mathbf{V}, [\rho]_{\vec{\omega}} \mathbf{V} \rangle \ .$$

Theorem 0.13. *Let $\rho(t)$ be a differentiable path in $\mathfrak{S}_+$ defined on $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ such that $\rho(0) = \rho_0$. Then there is a unique vector field $\mathbf{V} \in \bigoplus^{|\mathcal{J}|}\mathcal{A}$ of the form $\mathbf{V} = \nabla U$ with $U \in \mathcal{A}$, for which the non-commutative continuity equation*

$$\dot{\rho}(0) = -\operatorname{div}([\rho_0]_{\vec{\omega}}\mathbf{V}) = -\operatorname{div}([\rho_0]_{\vec{\omega}}\nabla U) \qquad (1)$$

holds. Moreover, U can be taken to be traceless, and is then self-adjoint. Furthermore, if $\mathbf{W}$ is any other vector field such that $\dot{\rho}(0) = -\nabla^\dagger([\rho_0]_{\vec{\omega}}\mathbf{W})$, then

$$\|\mathbf{V}\|_{\sigma,\rho_0} < \|\mathbf{W}\|_{\sigma,\rho_0} \ .$$

Definition 0.14. *For each $\rho \in \mathfrak{S}_+$, we identify the tangent space T_ρ at ρ, with the set of gradients vector fields $\{\nabla U \ : \ U \in \mathcal{A}\ ,\ U = U^*\}$. We define the Riemannian metric g_σ on $\mathfrak{S}_+$ by*

$$\|\dot{\rho}(0)\|^2_{g_{\sigma(\rho(0))}} = \|\mathbf{V}\|^2_{\sigma,\rho(0)}$$

where $\dot{\rho}(0)$ and $\mathbf{V}$ are related by (1). If $\mathcal{F}$ is any differentiable function on $\mathfrak{S}_+$, the corresponding gradient vector field, denoted $\mathrm{grad}_{g_\sigma}\mathcal{F}(\rho)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\rho(t))\bigg|_{t=0} = g_\sigma\big(\dot{\rho}(0), \mathrm{grad}_{g_\sigma}\mathcal{F}(\rho)\big)$$

for all differentiable paths $\rho(t)$ defined on $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ with $\rho(0) = \rho$.

Let $\frac{\delta \mathcal{F}}{\delta \rho}(\rho)$ denote the derivative of $\mathcal{F}$: For all self-adjoint $A \in \mathcal{A}$,

$$\lim_{t \to 0} \frac{1}{t}(\mathcal{F}(\rho + tA) - \mathcal{F}(\rho) = \mathrm{Tr}\left[\frac{\delta \mathcal{F}}{\delta \rho}(\rho) A\right] .$$

In particular, when

$$\dot{\rho}(0) + \mathrm{div}([\rho_0]_{\vec{\omega}} \nabla U) = 0$$

is satisfied for some U,

$$\mathrm{Tr}\left[\frac{\delta \mathcal{F}}{\delta \rho}(\rho)\,\mathrm{div}([\rho]_{\vec{\omega}} \nabla U)\right] = -g_{\vec{\omega}}\left(\nabla \frac{\delta \mathcal{F}}{\delta \rho}(\rho), \nabla U\right) .$$

Theorem 0.15. *Let* $\mathscr{P}_t = e^{t\mathscr{L}}$ *be QMS on* $\mathcal{A}$ *that satisfies the* σ*-DBC for* $\sigma \in \mathfrak{S}_+(\mathcal{A})$*. Then*

$$\frac{\partial}{\partial t}\rho = \mathscr{L}^\dagger \rho \tag{2}$$

is gradient flow for the relative entropy $D(\cdot||\sigma)$ *in the metric* g_σ *canonically associated to* σ*.*